RELATIVE CHERN CHARACTER, BOUNDARIES AND INDEX FORMULÆ

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1.0; Revised: 5-15-2008; Run: August 1, 2008

Abstract. For three classes of elliptic pseudodifferential operators on a compact manifold with boundary which have 'geometric K-theory', namely the 'transmission algebra' introduced by Boutet de Monvel [5], the 'zero algebra' introduced by Mazzeo in [9, 10] and the 'scattering algebra' from [16] we give explicit formulæ for the Chern character of the index bundle in terms of the symbols (including normal operators at the boundary) of a Fredholm family of fibre operators. This involves appropriate descriptions, in each case, of the cohomology with compact supports in the interior of the total space of a vector bundle over a manifold with boundary in which the Chern character, mapping from the corresponding realization of K-theory, naturally takes values.

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Introduction

Among the different algebras of pseudodifferential operators on a compact manifold with boundary, those for which the stable homotopy classes of the Fredholm elements reduce to the relative K-theory of the cotangent bundle can be expected to have the simplest, most local, index formulæ. These cases include the algebra

of transmission operators introduced by Boutet de Monvel, for which this is shown in essence in [5], the algebra of 'zero pseudodifferential operators' of [11] and the algebra of 'scattering pseudodifferential operators' of [16]. For a Fredholm family of such operators defined on the fibers of a fibration,

$$\begin{tikzcd} Z \arrow[r, dash] & M \arrow[d, "\psi"] \\ & B, \end{tikzcd} \tag{1}$$

the families index, composed with the Chern character, therefore gives a map (the same in all three cases)

$$K^0\left(T^*(M^\circ/B)\right) \xrightarrow{\text{ind}} K(B) \xrightarrow{\text{Ch}} H^{\text{even}}(B),\ M^\circ = M \setminus \partial M. \tag{2}$$

This map is always given by the general Atiyah-Singer formula (including here the extension by Atiyah and Bott) just as in the boundaryless case in [2, 3]

$$\operatorname{Ch}(\operatorname{Ind}(A)) = \psi_!\left(\operatorname{Ch}([A])\operatorname{Td}(Z)\right) \tag{3}$$

where $\operatorname{Ch}([A])$ is the Chern character,

$$\operatorname{Ch} : K^0\left(T^*(M^\circ/B)\right) \longrightarrow H_{\text{c}}^{\text{even}}\left(T^*(M^\circ/B)\right). \tag{4}$$

In fact, the identification, in the three cases, of the stable homotopy classes of the Fredholm elements with the relative K-theory involves non-trivial homotopies. As a result (3) is not really a 'formula' for the families index. Here we give much more explicit formulæ for the Chern character in terms of the 'symbolic' data determining the Fredholm condition for the family. In each case this corresponds to the ellipticity of the interior symbol, uniformly up to the boundary in an appropriate sense, together with the invertibility of a boundary family. To represent the Chern character we construct relative deRham chain complexes, all with cohomology $H_{\text{c}}^*(T^*(M^\circ/B))$, tailored to each setting and then construct Chern-Weil forms depending on the leading symbolic data. Then $\psi_!$ is the (generalized) integration map on cohomology from $T^*(M/B)$ to B.

To explain the strategy behind these explicit versions of the formulæ (3), consider the familiar case of an elliptic family of (classical) pseudodifferential operators $A \in \Psi^m(M/B;\mathbb{E})$ where $\mathbb{E} = (E^+, E^-)$ is a superbundle (i.e. a $\mathbb{Z}_2$-graded bundle) over M and the fibration (1) has compact boundaryless fibres. This is the families setting of Atiyah and Singer and the analytic index is a map as in (2), in this case the Chern character gives

$$\operatorname{Ch}(\operatorname{Ind}(A)) : K(T^*(M/B)) \longrightarrow H^{\text{even}}(B).$$

The K-class is fixed by the (invertible) symbol of A,

$$a = \sigma_m(A) \in \mathcal{C}^\infty(S^*(M/B); \hom(\mathbb{E}) \otimes N_m)$$

where N_m is a trivial real line bundle capturing the homogeneity. Fedosov in [6] gives an explicit formula for the Chern character of the compactly supported K-class determined by $\mathbb{E}$ and a as a deRham class with compact support on $T^*(M/B)$. Modifying his approach slightly we consider the representation of $H_{\text{c}}^*(W)$, for any real vector bundle over M, as the hypercohomology of the relative complex

$$\mathcal{C}^\infty(M;\Lambda^*) \oplus \mathcal{C}^\infty(\mathbb{S}W;\Lambda^*),\ D = \begin{pmatrix} d & \\ -\pi^* & -d \end{pmatrix} \tag{5}$$

where $\pi : W \longrightarrow M$ is the bundle projection and $\mathbb{S}W = (W \setminus O_M)/\mathbb{R}^+$ is the sphere bundle of M. The Chern-Weil formula given by Fedosov in terms of connections and curvatures, naturally fits into this representation as the class

(6)
$$\begin{gathered}\operatorname{Ch}([A]) = \operatorname{Ch}(\mathbb{E}, a) = \operatorname{Ch}(\mathbb{E}) \oplus \widetilde{\operatorname{Ch}}(a) \in \mathcal{C}^\infty(M; \Lambda^{\text{even}}) \oplus \mathcal{C}^\infty(S^*(M/B); \Lambda^{\text{odd}})\\ \operatorname{Ch}(\mathbb{E}) = \operatorname{tr} e^{\omega_+} - \operatorname{tr} e^{\omega_-},\\ \widetilde{\operatorname{Ch}}(a) = -\frac{1}{2\pi i}\int_0^1 \operatorname{tr}\left(a^{-1}(\nabla a)e^{w(t)}\right) dt \text{ where}\\ w(t) = (1-t)\omega_+ + ta^{-1}\omega_- a + \frac{1}{2\pi i}t(1-t)(a^{-1}\nabla a)^2.\end{gathered}$$

The push-forward map on cohomology, $\psi_!$, becomes integration of the second form in (5) and the Atiyah-Singer formula, (3), becomes an explict fibre integral

$$\operatorname{Ch}(\operatorname{Ind}(A)) = \int_{S^*(M/B)} \left(\widetilde{\operatorname{Ch}}(a) \operatorname{Td}(\psi)\right). \tag{7}$$

We proceed to discuss formulæ essentially as explicit as (6) and (7) for the Chern character of the index bundle for families of pseudodifferential operators on a manifold with boundary, where the uniformity conditions up to the boundary correspond to one of the three cases mentioned above.

The simplest of the three cases, really because it is the most commutative, corresponds to the scattering calculus of [16]. This arises geometrically in the context of asymptotically flat manifolds and in particular includes constant-coefficient differential operators on a vector space, thought of as acting on the radial compactification as a compact manifold with boundary. Thus, the fibration (1) now, and from now on, has fibres which are compact manifolds with boundary and we consider a 'fully elliptic' family $A \in \Psi^m_{\text{sc}}(M/B; \mathbb{E})$. As well as a uniform version of the symbol, $a = \sigma(A) \in \mathcal{C}^\infty({}^{\text{sc}}\mathrm{S}^*(M/B); \hom(\mathbb{E}) \otimes N_m)$ (acting on a boundary-rescaled version of the cosphere bundle) there is an invariantly-defined boundary symbol $b = \beta(A) \in \mathcal{C}^\infty(\overline{{}^{\text{sc}}\mathrm{T}^*_{\partial M}(M/B)}; \hom(\mathbb{E}) \otimes N_m)$. Together these two symbols form a smooth section of the bundle over the boundary of the radial compactification $\overline{{}^{\text{sc}}\mathrm{T}^*(M/B)}$ which is continuous at the corner. Full ellipticity of the family A reduces to invertibility of this joint symbol and this is equivalent to the requirement that A be a family of Fredholm operators on the natural geometric Sobolev spaces. Since $\overline{{}^{\text{sc}}\mathrm{T}^*(M/B)}$ is a topological manifold with boundary, these full symbols provide a chain space for the K-theory (with compact supports in the interior) and the analytic index becomes a well-defined map as in (2).

To give an explicit formula for the Chern character in this scattering setting we use a similar relative chain complex to (5), now associated to a vector bundle $\pi : W \longrightarrow M$ over a manifold with boundary. Namely,

(8)
$$\begin{gathered}\mathcal{C}^\infty(M; \Lambda^*) \oplus \left\{(u, v); u \in \mathcal{C}^\infty(\mathbb{S}W; \Lambda^*),\ v \in \mathcal{C}^\infty(\overline{W}_{\partial M}; \Lambda^*) \text{ and } \iota_\partial^* u = \iota_\partial^* v\right\},\\ D = \begin{pmatrix} d & 0 \\ \phi^* & -d \end{pmatrix},\ \phi = \begin{pmatrix} -\pi^* \\ -\pi^*\iota_\partial^* \end{pmatrix}.\end{gathered}$$

The cohomology of this chain complex is $H^*_{\text{c}}(W^\circ)$, the compactly supported cohomology of W restricted to the interior of M, and the Chern character is represented by the explicit forms

$$\operatorname{Ch}([A]) = \operatorname{Ch}(\mathbb{E}, a, b) = \operatorname{Ch}(\mathbb{E}) \oplus (\widetilde{\operatorname{Ch}}(a), \widetilde{\operatorname{Ch}}(b)) \tag{9}$$

where $\widetilde{\mathrm{Ch}}(a)$ and $\widetilde{\mathrm{Ch}}(b)$ are given by Fedosov's formula (6). The Atiyah-Singer formula (3) then follows and in this case it becomes quite clear that the boundary symbol is a rather complete analogue of the usual symbol and enters into the index formula in the same way.

The other two contexts in which we give such an explicit index formula are similar, but more complicated and less symmetric between the boundary and the cosphere bundle (the high-momentum limit) because there is more residual non-commutativity at the boundary. Note that we are only considering those settings in which the analytic index gives a map as in (2) (always the same map of course!) This restriction corresponds to the fact, as we shall see, that the only non-commutativity which remains at the boundary is in the normal variables – tangential non-commutativity, as occurs in the cusp or b-calculi, induces an analytic index map from a different K-theory in place of the relative K-theory in (2) and this leads to more subtlety in the index formula.

Consider next the 'zero calculus' which corresponds geometrically to asymptotically hyperbolic (or 'conformally compact') manifolds and is so named because it quantizates the Lie algebra of vector fields which vanish at the boundary of any compact manifold with boundary (as opposed to the scattering calculus which quantizes the smaller Lie algebra in which the normal part also vanishes to second order). As noted this Lie algebra is not commutative at the boundary, rather it is solvable with tangential part forming an Abelian subalgebra on which the normal part acts by homotheity. As a result the final formula has a truly regularized Chern character, an eta form, coming from the normal part. Again we consider a family of pseudodifferential operators, now $A \in \Psi_0^m(M/B;\mathbb{E})$ for a fibration (1) with fibres compact manifolds with boundary modelled on Z. The 'fully ellipticity' of such a family, corresponding to its being Fredholm on the natural 'zero' Sobolev spaces, reduces to the invertibility of the (uniform) symbol, $a = \sigma_m(A) \in \mathcal{C}^\infty({}^0S^*(M/B);\hom(\mathbb{E}) \otimes N_m)$ together with the invertibility of the normal operator, or equivalently the reduced normal family. The former takes values in the bundle, over the boundary, of invariant pseudodifferential operators on the Lie group associated to the tangent solvable Lie algebra mentioned above. The condition here is invertibility on the appropriate Sobolev spaces since this inverse, because of the appearance of non-trivial asymptotic terms, lies in a larger space of pseudofferential operators on the group. The reduced normal operator, $\mathrm{RN}(A)$, corresponds to decomposition of the normal operator in terms of the representations of the solvable group.

In the case of the zero calculus it is less obvious, but shown in [1], that this symbolic data, the invertibility of which fixes the Fredholm property of A, also gives a chain space for the relative K-theory $[A] = [(a, \mathrm{RN}(A))] \in K(T^*(M/B))$. Similarly, the chain complex leading again to cohomology with compact supports is more involved and depends on more of the structure of the zero cotangent bundle. More abstractly, consider a real vector bundle W over M with restriction to the boundary having a trivial real line subbundle $L \subset W_{\partial M}$, set $U = W_{\partial M}/L$ and

$$
\begin{gathered}
\mathcal{C}^\infty(M;\Lambda^k) \oplus \mathcal{C}^\infty(\mathbb{S}W \cup \overline{L};\Lambda^{k-1}) \oplus \mathcal{C}^\infty(\mathbb{S}U;\Lambda^{k-3}),\\
\mathcal{C}^\infty(\mathbb{S}W \cup \overline{L};\Lambda^{k-1}) = \left\{(a,\gamma) \in \mathcal{C}^\infty(\mathbb{S}W;\Lambda^{k-1}) \oplus \mathcal{C}^\infty(\overline{L};\Lambda^{k-1}); i_\pm^* a = i_\pm^* \gamma\right\}\\
D = \begin{pmatrix} d & 0 & 0 \\ \phi_1 & -d & 0 \\ 0 & \phi_2 & d \end{pmatrix},\ \phi_1 = \begin{pmatrix} -\pi_{\mathbb{S}W}^* \\ -\pi_{\overline{L}}^* \iota_\partial^* \end{pmatrix},\ \phi_2 = \begin{pmatrix} -\nu_*^{\mathbb{S}W}, \pi_{\mathbb{S}U}^* \nu_*^{\overline{L}} \end{pmatrix}.
\end{gathered}
\tag{10}
$$

Here $i_\pm$ are the restrictions to the two boundary components of $\overline{L}$ (which are also submanifolds of $\mathbb{S}W$), the maps π are the various bundle projections and $\nu_*^{\mathbb{S}W}$, ν_*^L are (well-defined) push-forward maps along the fibres of L, the first from $\partial\mathbb{S}W$ to $\mathbb{S}U$ and the second from $\overline{L}$ to ∂M. As already anticipated the cohomology of this complex is canonically isomorphic to the compactly supported cohomology of W over M°.

The reduced normal operator for the zero calculus is a family of pseudodiferential operators on an interval, although with different uniformity behaviour at the two ends. It is fixed by the choice of a splitting of the zero cotangent bundle over the boundary and the choice of a metric and is then naturally parametrized by $S^*(\partial M/B)$ which is $\mathbb{S}U$ in the preceeding paragraph. In terms of the representation of the cohomology with compact supports of $T^*(M^\circ/B)$ given by (10), with $W = {}^0T^*(M/B)$, the Chern character is given by the explicit forms

(11)
$$\mathrm{Ch}([A]) = \mathrm{Ch}(a, \mathrm{RN}(A)) = \mathrm{Ch}(\mathbb{E}) \oplus \left(\widetilde{\mathrm{Ch}}(a), \widetilde{\mathrm{Ch}}(I_b(\mathrm{RN}(A)))\right) \oplus (-\eta(\mathrm{RN}(A))),$$

where $I_b(\mathrm{RN}(A))$ is the model operator of the reduced normal operator at one end of the interval, known as the indicial family, and $\eta(\mathcal{N})$ is given by an expression similar to that defining $\widetilde{\mathrm{Ch}}(a)$ but taking into account that the operators involved are not trace-class and do not commute, namely

$$-\frac{1}{2\pi i}\int_0^1 (1-t)\,{}^R\mathrm{Tr}_{\mathrm{b,sc}}\left(\mathcal{N}^{-1}\,(\nabla\mathcal{N})\,e^{\omega_{\mathcal{N}}(t)}\right) + t\,{}^R\mathrm{Tr}_{\mathrm{b,sc}}\left((\nabla\mathcal{N})\,e^{\omega_{\mathcal{N}}(t)}\mathcal{N}^{-1}\right)\,dt.$$

The renormalized trace appearing in this formula is defined in Appendix B following previous work of the second author and Victor Nistor [18]. As in the scattering case we obtain an explicit representative of the Chern character.

Finally, we consider the calculus of Boutet de Monvel, the 'transmission calculus', which contains classical elliptic boundary value problems and their parametrices. Geometrically this calculus corresponds to Riemannian manifolds with boundary and so the connection to relative K-theory is immediate and already present in [5]. An element of the transmission calculus is represented by a matrix of operators

$$\mathcal{A} = \begin{pmatrix} \gamma^+ A + B & K \\ & \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty(X; E^+) \\ \oplus \\ \mathcal{C}^\infty(\partial X; F^+) \end{matrix} \longrightarrow \begin{matrix} \mathcal{C}^\infty(X; E^-) \\ \oplus \\ \mathcal{C}^\infty(\partial X; F^-) \end{matrix}$$

acting on the superbundles $\mathbb{E}$ over X and $\mathbb{F}$ over ∂X. Whether or not a family $\mathcal{A} \in \Psi^m_{\mathrm{tm}}(M/B; \mathbb{E}, \mathbb{F})$ is Fredholm is again determined by invertibility of two model operators, the interior symbol and the boundary symbol. An operator is elliptic if the former is invertible and 'fully elliptic' if they are both invertible. Similarly to the zero calculus, the model operator at the boundary is a family of pseudodifferential operators parametrized by the cosphere bundle of the boundary, but here these operators are of Wiener-Hopf type.

As before there is a convenient description of the relative cohomology of the cotangent bundle over the interior that is compatible with the represention of a K-class by a fully elliptic family of transmission operators. For this consider W, L, and U as described in (10) above, together with a restricted space of sections of $\mathbb{S}W$,

$$\mathcal{C}^\infty_\pm(\mathbb{S}W; \Lambda^*) = \{\alpha \in \mathcal{C}^\infty(\mathbb{S}W; \Lambda^*) : i^*_{L^+}\alpha = i^*_{L^-}\alpha,\ i^*_{L^+}d\alpha = i^*_{L^-}d\alpha\},$$

and form the complex

$$\mathcal{C}^\infty(M;\Lambda^k)\oplus\left(\mathcal{C}^\infty_\pm(\mathbb{S}W;\Lambda^{k-1})\oplus\mathcal{C}^\infty(\partial M;\Lambda^{k-2})\right)\oplus\mathcal{C}^\infty(\mathbb{S}U;\Lambda^{k-3}), \tag{12}$$
$$D=\begin{pmatrix} d & 0 & 0\\ \phi_1 & -d & 0\\ 0 & \phi_2 & d\end{pmatrix},\ \phi_1=\begin{pmatrix}-\pi^*_{\mathbb{S}W}\\ -i^*_{\partial M}\end{pmatrix},\ \phi_2=\left(\nu^{\mathbb{S}W}_*,-\pi^*_{\mathbb{S}U}\right).$$

with the notation as in the previous paragraph. The cohomology of this complex is again canonically isomorphic to the compactly supported cohomology of W over M°.

If a and N are respectively the interior symbol and boundary symbol of a fully elliptic family of transmission operators, then the Chern character of the associated K-theory class is represented by

$$\mathrm{Ch}(\mathbb{E},\mathbb{F},a,N)=\left(\mathrm{Ch}(\mathbb{E}),\left(\widetilde{\mathrm{Ch}}(a),-\mathrm{Ch}(\mathbb{E}_\partial\oplus\mathbb{F})\right),-\eta(N)\right)$$

in the complex (12) with W, L, and U again equal to $T^*(M/B)$, the normal bundle to the boundary, and $T^*(\partial M/B)$ respectively. The form $\eta(N)$ is formally the same as in the zero calculus, but the renormalization of the trace is done in a different fashion, due to Fedosov (see (3.25) below).

As already mentioned, central to this discussion is the fact that the K-theory described by fully elliptic families in these cacluli is the topological K-theory of the cotangent bundle in the interior. A well-known consequence is that quantization of an elliptic symbol to a Fredholm operator is only possible if the 'Atiyah-Bott obstruction' of the symbol vanish. One way around this is to quantize into other pseudodifferential calculi. For instance, the b-calculus, which is well-adapted to manifolds with asymptotically cylindrical ends, is *universal* in the sense that any elliptic symbol can be quantized to a Fredholm operator. Whereas the calculi described above are asymptotically non-commutative in (at most) the normal direction, the b calculus is in the same sense asymptotically non-commutative in all directions. This is related to the fact that the eta invariants described above are 'local in the boundary and global in the normal direction to the boundary' while the eta invariant in the Atiyah-Patodi-Singer index theorem is global in the boundary. An analysis of the Chern character for the b-calculus and related calculi is the subject of an ongoing project of the second author with Frédéric Rochon [19].

This manuscript is divided into three parts. Section 1 is devoted to a discussion of cohomology. We describe the approach to relative cohomology that we will follow together with some standard properties and work out various ways of representing $H^*_c(T^*M^\circ/B)$. In section 2 the represention of a class in $K_{\mathrm{c}}(T^*M^\circ/B)$ by a family of fully elliptic operators in either the scattering, zero, or transmission calculus is recalled. Finally, in section 3, we put these discussions together and obtain explicit formulæfor the Chern character

$$\mathrm{Ch}:K_{\mathrm{c}}(T^*M^\circ/B)\longrightarrow H^{\mathrm{even}}_{\mathrm{c}}(T^*M^\circ/B)$$

as described above. In each case an explicit 'Atiyah-Singer' formula for the Chern character of the index bundle is given using only the appropriate model operators.

1. Relative cohomology

Suppose that $(\mathcal{C}^*_i,d)$, $i=1,\dots,N$, are $\mathbb{Z}$-graded differential complexes and $\phi_i:\mathcal{C}^k_i\longrightarrow\mathcal{C}^{k+1-f_i}_{i+1}$ for $1\le i<N$, is a complex of chain maps between them, so

$d\phi_i = \phi_{i+1}d$. Then the complexes can be 'rolled up' into one complex. In fact only the cases $N = 2$ and $N = 3$ arise here, so consider first $N = 2$:

$$
\begin{tikzcd}
\cdots \arrow[r, "d"] & \mathcal{C}_1^{k-1} \arrow[r, "d"] \arrow[d, "\phi"] & \mathcal{C}_1^{k} \arrow[r, "d"] \arrow[d, "\phi"] & \mathcal{C}_1^{k+1} \arrow[r, "d"] \arrow[d, "\phi"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{C}_2^{k-f} \arrow[r, "d"] & \mathcal{C}_2^{k+1-f} \arrow[r, "d"] & \mathcal{C}_2^{k+2-f} \arrow[r, "d"] & \cdots .
\end{tikzcd} \tag{1.1}
$$

Then ϕ induces a map of the corresponding hypercohomologies which we can also denote $\phi : \mathcal{H}_1^k \longrightarrow \mathcal{H}_2^{k+1-f}$. The relative chain complex

$$(\mathcal{C}^k, D),\ \mathcal{C}^k = \mathcal{C}_1^k \oplus \mathcal{C}_2^{k-f},\ D = \begin{pmatrix} d & 0 \\ \phi & -d \end{pmatrix} \tag{1.2}$$

is such that inclusion and projection give an exact sequence

$$\mathcal{C}_2^{k-f} \xrightarrow{\iota} \mathcal{C}^k \xrightarrow{p} \mathcal{C}_1^k. \tag{1.3}$$

The hypercohomology of (1.2), denoted $\mathcal{H}^*(\mathcal{C}_*, \phi)$, may be computed from a spectral sequence, in this case a rather simple one corresponding to the fact that the long exact sequence associated to (1.3) is

$$\cdots \longrightarrow \mathcal{H}_2^{k-f} \xrightarrow{\iota} \mathcal{H}^k(\mathcal{C}_*, \phi) \xrightarrow{p} \mathcal{H}_1^k \xrightarrow{\phi} \mathcal{H}_2^{k-f} \cdots . \tag{1.4}$$

For $N = 3$ the ϕ_i give a commutative diagram

$$
\begin{tikzcd}
\cdots \arrow[r, "d"] & \mathcal{C}_1^{k-1} \arrow[r, "d"] \arrow[d, "\phi_1"] & \mathcal{C}_1^{k} \arrow[r, "d"] \arrow[d, "\phi_1"] & \mathcal{C}_1^{k+1} \arrow[r, "d"] \arrow[d, "\phi_1"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{C}_2^{k-f_1} \arrow[r, "d"] \arrow[d, "\phi_2"] & \mathcal{C}_2^{k+1-f_1} \arrow[r, "d"] \arrow[d, "\phi_2"] & \mathcal{C}_2^{k+2-f_1} \arrow[r, "d"] \arrow[d, "\phi_2"] & \cdots \\
\cdots \arrow[r, "d"] & \mathcal{C}_3^{k+1-f_1-f_2} \arrow[r, "d"] & \mathcal{C}_3^{k+2-f_1-f_2} \arrow[r, "d"] & \mathcal{C}_3^{k+3-f_1-f_2} \arrow[r, "d"] & \cdots
\end{tikzcd} \tag{1.5}
$$

The 'rolled up' double complex is

$$(\mathcal{C}^k, D),\ \mathcal{C}^k = \bigoplus_{i=1}^{3} \mathcal{C}_i^k,\ D = \begin{pmatrix} d & 0 & 0 \\ \phi_1 & -d & 0 \\ 0 & \phi_2 & d \end{pmatrix}. \tag{1.6}$$

Of course, the second two rows in (1.5) are an example of (1.1), so give the complex

$$(\widetilde{\mathcal{C}}_2, \widetilde{d}),\ \widetilde{\mathcal{C}}_2^k = \mathcal{C}_2^k \oplus \mathcal{C}_3^{k-f_2},\ \widetilde{d} = \begin{pmatrix} d & 0 \\ \phi_2 & -d \end{pmatrix} \tag{1.7}$$

such that

$$\widetilde{\mathcal{C}}_2^{k-f_1} \xrightarrow{\iota} \mathcal{C}^k \xrightarrow{p} \mathcal{C}_1^k \tag{1.8}$$

is a short exact sequence. The hypercohomologies therefore give long exact sequences as in (1.4)

$$
\begin{gathered}
\cdots \longrightarrow \widetilde{\mathcal{H}}_2^{k-f} \xrightarrow{\iota} \mathcal{H}^k(\mathcal{C}_*, \phi) \xrightarrow{p} \mathcal{H}_1^k \xrightarrow{\phi_1} \widetilde{\mathcal{H}}_2^{k+1-f} \cdots \\
\cdots \longrightarrow \mathcal{H}_3^{k-f} \xrightarrow{\iota} \widetilde{\mathcal{H}}^k \xrightarrow{p} \mathcal{H}_2^k \xrightarrow{\phi_2} \mathcal{H}_3^{k+1-f} \cdots .
\end{gathered} \tag{1.9}
$$

Thus the case $N > 2$ reduces to an iteration of $N = 2$ cases.

The most obvious case of relative cohomology in a deRham setting arises from a smooth map between compact manifolds (possibly with corners)
(1.10)
$$\psi : M \longrightarrow X,\ \text{with } \mathcal{C}_2 = \mathcal{C}^\infty(M;\Lambda^*),\ \mathcal{C}_1 = \mathcal{C}^\infty(X,\Lambda^*) \text{ and } D = \begin{pmatrix} d & 0 \\ \psi^* & -d \end{pmatrix}.$$
In this case we may denote the relative cohomology by $\mathcal{H}^*(M,\psi)$. Note that this is precisely how relative cohomology is defined in [4, §6].

Several variants of relative cohomology, leading to standard cohomology groups, are of interest here, only the last (and most fundamental for the zero index formula) corresponds to $N=3$.

1.1. **Homotopy invariance and module structure.** In all of the cases we will consider below the complexes $\mathcal{C}_i^*$ will be built up from differential forms and will inherit more structure. We point out some of these properties for later use.

1.1.1. *Module structure.* First, suppose that each of the $\mathcal{C}_i$ has a graded product
$$\wedge_i : \mathcal{C}_i^j \times \mathcal{C}_i^k \to \mathcal{C}_i^{j+k} \text{ s.t.}$$
$$d\left(\alpha \wedge_i \beta\right) = d\alpha \wedge_i \beta + (-1)^{|\alpha|}\alpha \wedge_i d\beta,\ \phi_i\left(\alpha \wedge_i \beta\right) = \phi_i\alpha \wedge_{i+1} \phi_i\beta.$$

Lemma 1.1. *If $N=3$ and each $\mathcal{C}_i$ has a graded product as above, then $\mathcal{H}^*\left(\mathcal{C}_*,\phi\right)$ is a module over $\mathcal{H}^*\left(\mathcal{C}_1\right)$ through*
$$\mathcal{H}^*\left(\mathcal{C}_*,\phi\right) \times \mathcal{H}^*\left(\mathcal{C}_1\right) \ni \left(\alpha_i,\gamma\right) \stackrel{\wedge}{\longmapsto} \left(\alpha_1 \wedge_1 \gamma,\ \alpha_2 \wedge_2 \phi_1\gamma,\ \alpha_3 \wedge_3 \phi_2\phi_1\gamma\right) \in \mathcal{H}^*\left(\mathcal{C}_*,\phi\right))$$

Proof. Notice that if $(\alpha_i) \in \mathcal{C}_*^k$
$$D\left((\alpha_i)\wedge\gamma\right) = \begin{pmatrix} d\left(\alpha_1 \wedge_1 \gamma\right) \\ \phi_1\left(\alpha_1 \wedge_1 \gamma\right) - d\left(\alpha_2 \wedge_2 \phi_1\gamma\right) \\ \phi_2\left(\alpha_2 \wedge_2 \phi_1\gamma\right) + d\left(\alpha_3 \wedge_3 \phi_2\phi_1\gamma\right) \end{pmatrix}$$
$$= D\left(\alpha_i\right)\wedge\gamma + (-1)^k \begin{pmatrix} \alpha_1 \\ -\left(-1\right)^{-f_1}\alpha_2 \\ (-1)^{-f_1-f_2}\alpha_3 \end{pmatrix} \wedge d\gamma.$$
Thus if f_1 and f_2 are odd D satisfies a Leibnitz rule with respect to $\wedge$. In any case it is true that if γ is closed and (α_i) is D-closed then $(\alpha_i)\wedge\gamma$ is D-closed and represents a class in $\mathcal{H}^*\left(M,\phi\right)$ depending only on the class of γ in $\mathcal{H}^*\left(\mathcal{C}_1\right)$ and (α_i) in $\mathcal{H}^*\left(M,\phi\right)$. □

1.1.2. *Homotopy invariance.* Recall the usual proof of homotopy invariance of the Chern character on a closed manifold. A one-parameter family of connections on a fixed bundle over a space X, can be interpreted as a single connection on the same bundle pulled-back to $X\times[0,1]$. Since the Chern character of this connection is closed, the cohomology class of the Chern character of the family of connections is constant in the parameter. What is important here is that the space of forms on $X\times[0,1]_r$ is really two copies of the space of forms on X,
$$\Omega^*\left(X\times[0,1]_r\right) \cong \Omega^*X \oplus dr\wedge\Omega^{*-1}X$$
and that the differential becomes $\begin{pmatrix} d & \\ \partial_r & -d \end{pmatrix}$ with respect to this splitting.

The spaces $\mathcal{C}_i$ below will generally be direct sums of spaces of differential forms with perhaps some compatibility conditions. In order to carry out the classical

argument for the homotopy invariance of the Chern character, assume that a one-parameter family of elements $(\alpha(r))$ of $\mathcal{C}_*$ define an element of $\widetilde{\mathcal{C}}_*$ with

$$\widetilde{\mathcal{C}}_*^i = \mathcal{C}_*^i \oplus \mathcal{C}_{*-1}^i \quad \left(= \mathcal{C}_*^i + dr \wedge \mathcal{C}_{*-1}^i\right) \tag{1.11}$$

and that the action of d and ϕ_i is extended to $\widetilde{\mathcal{C}}_*$ so that with respect to this splitting

$$d = \begin{pmatrix} d & \\ \partial_r & -d \end{pmatrix} \text{ and } \phi_i = \begin{pmatrix} \phi_i & \\ & \phi_i \end{pmatrix}.$$

Then D acts on $\widetilde{\mathcal{C}}_* = \mathcal{C}_* \oplus \mathcal{C}_{*-1}$ by

$$\widetilde{D} = \begin{pmatrix} d & & & & & \\ \phi_1 & -d & & & & \\ & \phi_2 & d & & & \\ \partial_r & & & -d & & \\ & -\partial_r & & \phi_1 & d & \\ & & \partial_r & & \phi_2 & -d \end{pmatrix}$$

Lemma 1.2. *If the spaces $\mathcal{C}_*^i$ have the properties of differential forms as described above and $(\widetilde{\alpha}(r))$ is a $\widetilde{D}$ closed element of $\widetilde{\mathcal{C}}_*$ then $\widetilde{\alpha}(0)$ and $\widetilde{\alpha}(1)$ are cohomologous in $\mathcal{C}_*$.*

Proof. Write

$$\widetilde{\alpha}^i = \left(\alpha_t^i, \alpha_n^i\right) \quad \left(= \alpha_t^i + dr \wedge \alpha_n^i\right)$$

with respect to the splitting (1.11). Then the fact that $(\widetilde{\alpha})$ is $\widetilde{D}$-closed implies that

$$\begin{aligned} i_0^*\widetilde{\alpha} - i_1^*\widetilde{\alpha} = \int_0^1 \partial_r(\alpha_t)\ dr &= \int_0^1 \left(d\alpha_n^1, \phi_1\alpha_n^1 - d\alpha_n^2, -\phi_2\alpha_n^2 + d\alpha_n^3\right)\ dr \\ &= D\left(\int_0^1 \left(\alpha_n^1, -\alpha_n^2, \alpha_n^3\right)\ dr\right), \end{aligned}$$

hence $i_0^*\widetilde{\alpha}$ and $i_1^*\widetilde{\alpha}$ represent the same class in the cohomology of $(\mathcal{C}_*, D)$. □

1.1.3. *Push-forward.* Notice that, if M is oriented and $\dim Y < \dim M$, there is a well-defined integral

$$\int_M : \mathcal{H}(M, \psi) \longrightarrow \mathbb{C}. \tag{1.12}$$

Indeed, two representatives of the same class in $\mathcal{H}(M, \psi)$ differ an element in the image of D, i.e. of the form $(u, \psi^*u - dv)$, but $\int_M dv = 0$ by Stoke's theorem and $\int_M \psi^*u = 0$ for dimensional reasons, so the value of the integral is not affected.

More generally, if we have a pair $(\mathcal{C}_1^* \oplus \mathcal{C}_2^*)$, $(\widetilde{\mathcal{C}}_1^* \oplus \widetilde{\mathcal{C}}_2^*)$ of $\mathbb{Z}$-graded complexes with chain maps ϕ, $\widetilde{\phi}$ as in (1.2), and maps between them

$$\mathcal{C}_i^* \xrightarrow{\psi_i} \widetilde{\mathcal{C}}_i^{*-\ell}$$

satisfying $d\psi_i = \psi_i d$ and fitting into the commutative diagram

$$\begin{tikzcd} \mathcal{C}_1^k \arrow[r, "\phi"] \arrow[d, "\psi_1"] & \mathcal{C}_2^{k+1-f} \arrow[d, "\psi_2"] \\ \widetilde{\mathcal{C}}_1^{k-\ell} \arrow[r, "\widetilde{\phi}"] & \widetilde{\mathcal{C}}_2^{k-\ell+1-f} \end{tikzcd} \tag{1.13}$$

we get a map

$$\mathcal{H}^*\left(\mathcal{C}_*, \phi\right) \xrightarrow{\psi} \mathcal{H}^{*-\ell}\left(\widetilde{\mathcal{C}}_*, \widetilde{\phi}\right). \tag{1.14}$$

This situation occurs for instance for pull-back diagrams: Assume $X \xrightarrow{f} Z$ is a fibration of oriented manifolds and $E \xrightarrow{\pi} Z$ is a vector bundle, then we have

$$\begin{tikzcd} f^*\mathbb{S}E \arrow[r, "\widetilde{\pi}"] \arrow[d, "\widetilde{f}"] & X \arrow[d, "f"] \\ \mathbb{S}E \arrow[r, "\pi"] & Z \end{tikzcd} \quad \Longrightarrow \quad \begin{tikzcd} \Omega^*\left(f^*\mathbb{S}E\right) \arrow[d, "\widetilde{f}_*"] & \Omega^* X \arrow[l, "\widetilde{\pi}^*"'] \arrow[d, "f_*"] \\ \Omega^{*-\ell}\left(\mathbb{S}E\right) & \Omega^{*-\ell} Z \arrow[l, "\pi^*"'] \end{tikzcd}$$

where $\ell = \dim\left(X/Z\right)$ and f_* denotes the push-forward of differential forms by 'integration along the fibers'. Then (1.13) commutes (essentially because the fibers of f and $\widetilde{f}$ coincide) and we get a map $\mathcal{H}^*\left(f^*\mathbb{S}E, \widetilde{\pi}\right) \to \mathcal{H}^*\left(\mathbb{S}E, \pi\right)$ induced by f via (1.14), which we can call push-forward by f.

Alternately, consider a smooth fibration $N \xrightarrow{h} \Gamma$ with $\dim N > \dim \Gamma$ and the vertical cosphere bundle $\mathbb{S}^*\left(N/\Gamma\right) \xrightarrow{\pi} N$. We can use (1.14) to get a map

$$\mathcal{H}^*\left(\mathbb{S}^*\left(N/\Gamma\right), \pi\right) \xrightarrow{h_*} H^*\left(Y\right) \tag{1.15}$$

by taking $\ell = 2\dim\left(N/\Gamma\right) - 1$, $\widetilde{\mathcal{C}}_1^* = 0$, $\widetilde{\mathcal{C}}_2^* = \Omega^* Y$ and mapping between $\Omega^*\mathbb{S}^*\left(N/\Gamma\right)$ and $\Omega^* Y$ by the push-forward of forms along the map $h \circ \pi$. In this case (1.13) commutes because forms pulled back from N push-foward to zero along $h \circ \pi$. This push-forward map will be used in the formula for the families index theorem below. (Note that (1.12) is a particular case with $\Gamma = \{pt\}$).

1.2. **Manifold with boundary.**

A standard case of (1.10) arises when X is a compact manifold with boundary and ψ is the inclusion map for the boundary. Thus $\mathcal{C}_1 = \mathcal{C}^\infty(X; \Lambda^*)$ and $\psi = \iota_\partial : \partial X \hookrightarrow X$, $\mathcal{C}_2 = \mathcal{C}^\infty(\partial X; \Lambda^*)$. Then

Lemma 1.3. *For $\iota : \partial X \longrightarrow X$ the inclusion of the boundary of a compact manifold with boundary and*

$$\begin{gathered} \mathcal{C}_*^k = \mathcal{C}^\infty(X; \Lambda^k) \oplus \mathcal{C}^\infty(\partial X; \Lambda^{k-1}), \\ \mathcal{H}^k(\mathcal{C}_*, \phi) = \mathcal{H}^k(\partial X, \iota) = H^k(X; \partial X) = H_c^k(X \setminus \partial X) \end{gathered} \tag{1.16}$$

is relative cohomology in the usual sense.

Proof. This is completely standard but a brief proof is included for the sake of completeness and for later generalization.

If $(u, v) \in \mathcal{C}_k$ satisfies $D\left(u, v\right) = 0$ then there is $T\left(u, v\right) \in \mathcal{C}_{k-1}$ such that $DT\left(u, v\right) - (u, v) = (\omega, 0)$ with

$$\omega \in \mathcal{C}_c^\infty(X^\circ; \Lambda^k). \tag{1.17}$$

Taking a product neighbourhood of the boundary, $U = [0, 1)_x \times \partial X$, and denoting $Y = \partial X$ a smooth form decomposes on U as

$$u = u_t(x) + dx \wedge u_n(x),\ u_t \in \mathcal{C}^\infty([0, 1) \times Y; \Lambda^k Y),\ u_n \in \mathcal{C}^\infty([0, 1) \times Y; \Lambda^{k-1} Y) \tag{1.18}$$

with differential

$$du = d_Y u_t + dx \wedge \left(\frac{\partial}{\partial x} u_t(x) - d_Y u_n(x)\right). \tag{1.19}$$

So if (u,v) satisfies $D(u,v)=0$,

$$d_Y u_t = 0,\ \frac{\partial}{\partial x}u_t(x) - d_Y u_n(x) = 0 \text{ in } x<1,\ dv = u_t(0). \tag{1.20}$$

Choosing a cutoff function $\rho \in \mathcal{C}^\infty([0,1])$ with $\rho(x)=1$ in $x<\frac{1}{2}$, $\rho(x)=0$ in $x>\frac{3}{4}$, consider

$$T(u,v) = \left(\rho(x)\int_0^x u_n(s)\,ds + \rho(x)v, 0\right). \tag{1.21}$$

This satisfies $DT(u,v) = (u+\omega, v)$ where $\omega \in \mathcal{C}^\infty(X;\Lambda^k)$ has support away from the boundary as required in (1.17). Thus the complex retracts to the subcomplex of deRham forms with compact support in the interior and the cohomology is therefore the cohomology of X relative to its boundary. □

Corollary 1.4. *A particular case of §1.10 arises with $X=\overline{U}$, the radial compactification of a real vector bundle U over a compact manifold Y and $\partial X = \mathbb{S}U = (U\setminus 0_Y)/\mathbb{R}^+$ the sphere bundle, so*

$$\mathcal{H}^*(\partial X,\iota) = H^*_c(U). \tag{1.22}$$

1.3. **Sphere bundle of a real vector bundle.** Although Corollary 1.4 is a 'compact' representation of the compactly-supported cohomology of a real vector bundle over a manifold it is not the most natural one for index theory. In view of the contractibility of the fibres, instead of the inclusion of the sphere bundle as the boundary of the radial compactification of W we may consider instead simply the projection

$$\pi : \mathbb{S}W \longrightarrow X. \tag{1.23}$$

Denote by $\mathcal{H}^k(\mathbb{S}W,\pi)$ the cohomology of the complex

$$\mathcal{C}^\infty(X;\Lambda^*) \oplus \mathcal{C}^\infty(\mathbb{S}W;\Lambda^{*-1}),\quad D = \begin{pmatrix} d & 0 \\ -\pi^* & -d\end{pmatrix} \tag{1.24}$$

(We get the same cohomology with π^* instead of $-\pi^*$, but the latter leads to better signs in the expressions for the Chern characters below.)

Lemma 1.5. *For any real vector bundle over a compact manifold without boundary,*

$$\mathcal{H}^k(\mathbb{S}W,\pi) \cong H^k_c(W).$$

Proof. Recall that the cohomology with compact supports of W may be represented by the deRham cohomology of smooth forms with compact support on W. Let $i_0 : X \hookrightarrow W$ be the inclusion of the zero section and choose a metric on W so as to have a product decomposition

$$W \setminus i_0(X) = \mathbb{S}W \times \mathbb{R}^+ \tag{1.25}$$

and denote the projection onto the left factor by R.

Given a closed k-form u on W with compact support consider the map

$$\Phi(u) = \left(i_0^* u, (-1)^{k-1}R_* u\right) \in \mathcal{C}^\infty(X;\Lambda^k) \oplus \mathcal{C}^\infty(\mathbb{S}W;\Lambda^{k-1}). \tag{1.26}$$

We can think of introducing polar coordinates as pulling-back to the space $\mathbb{S}W\times\mathbb{R}^+$, say via a map β, in terms of which we have

$$\beta^* u = u_t + u_n \wedge dr, \text{ with } \mathrm{i}_{\partial_r} u_t = 0$$
$$\text{so } \Phi(u) = \left(u_t(0), (-1)^{k-1} \int_0^\infty u_n \, dr \right)$$

where we denote the interior product with ∂_r by i_{∂_r}. Notice that

$$\begin{aligned}\Phi(dv) &= \left(d_X i_0^* v, (-1)^k R_* \left[(d_{\mathbb{S}W} v_n + (-1)^k \partial_r v_t) \wedge dr \right] \right) \\ &= \left(d_X i_0^* v, -\pi^* i_0^* v - d_{\mathbb{S}W}[(-1)^{k-1} R_* v] \right) = D\Phi(v),\end{aligned}$$

so Φ defines a map on cohomology which is easily seen to be an isomorphism (for instance by using the commutative diagram

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-1}(\mathbb{S}W) \arrow[r] & \mathcal{H}^k(\mathbb{S}W,\pi) \arrow[r] & H^k(X) \arrow[r, "\pi^*"] & H^k(\mathbb{S}W) \arrow[r] & \cdots \\
\cdots \arrow[r] & H^{k-1}(\mathbb{S}W) \arrow[r] \arrow[u, "\mathrm{id}"] & H^k_c(W) \arrow[r] \arrow[u, "\Phi"] & H^k(\overline{W}) \arrow[r, "\pi^*"] \arrow[u, "i_0^*"] & H^k(\mathbb{S}W) \arrow[r] \arrow[u, "\mathrm{id}"] & \cdots
\end{tikzcd}$$

and the Five Lemma). □

1.4. **Bundle with line subbundle.** A variant of the setting of Lemma 1.5 arises in the index formula for perturbations of the identity in the zero algebra. There a real vector bundle, $W \longrightarrow Y$, has a trivial line subbundle $L \subset W$. Setting $U = W/L$

$$\mathcal{H}^*(\mathbb{S}U,\pi) = H^*_{\mathrm{c}}(U) = H^{*+1}_{\mathrm{c}}(W) \tag{1.27}$$

since $W \simeq U \oplus L$ with L by assumption trivial. For our purposes there is another more useful complex giving the same cohomology.

Consider the radial compactification of L, $\overline{L}$, obtained by attaching to each fiber of L the points at $\pm\infty$, L^+, L^-, and the forms

$$\mathcal{C}^\infty_\pm(\overline{L};\Lambda^*) = \{\alpha \in \mathcal{C}^\infty(\overline{L};\Lambda^*) : i^*_{L^+}\alpha = i^*_{L^-}\alpha,\ i^*_{L^+}d\alpha = i^*_{L^-}d\alpha\}.$$

We use the deRham complexes $\mathcal{C}^k_1 = \mathcal{C}^\infty_\pm(\overline{L};\Lambda^k)$ and $\mathcal{C}^k_2 = \mathcal{C}^\infty(\mathbb{S}U;\Lambda^k)$ with the chain map

$$-\pi^*\nu^L_* : \mathcal{C}^k_1 \longrightarrow \mathcal{C}^{k-1}_2,$$

(where $\nu^L_* : \mathcal{C}^\infty_\pm(\overline{L};\Lambda^*) \longrightarrow \mathcal{C}^\infty(Y;\Lambda^{*-1})$ is push-forward under $\pi_{\mathbb{S}W}$ restricted to L) to form the complex $\mathcal{C}^k = \mathcal{C}^k_1 \oplus \mathcal{C}^{k-2}_2$ (note $f=2$) with differential $\begin{pmatrix} d & 0 \\ -\pi^*\nu^L_* & -d \end{pmatrix}$. The cohomology of this complex will be denoted $\mathcal{H}^*\left(\mathbb{S}U,\overline{L}\right)$. Notice that for $\alpha \in \mathcal{C}^\infty_\pm(\overline{L};\Lambda^*)$ we have $d\nu^L_*\alpha = \nu^L_* d\alpha$ (cf. Lemma 1.8 below).

Lemma 1.6. *There are natural isomorphisms in cohomology*

$$\mathcal{H}^k(\mathcal{C}, -\pi^*\nu^L_*) = H^k_c(W) = \mathcal{H}^{k-1}(\mathbb{S}U,\pi). \tag{1.28}$$

Proof. We have the following commutative diagram relating the long exact sequences for $\mathcal{H}^*\left(\mathbb{S}U, \overline{L}\right)$ and $\mathcal{H}^*\left(\mathbb{S}U, \pi\right)$ described in (1.4),

$$\begin{tikzcd}
\cdots \arrow[r] & H^{k-2}(\mathbb{S}U) \arrow[r] \arrow[d, "\mathrm{id}"] & \mathcal{H}^k\left(\mathbb{S}U, \overline{L}\right) \arrow[r] \arrow[d, "\nu_*^{\overline{L}}"] & H^k\left(\overline{L}\right) \arrow[r, "-\pi^*\nu_*^L"] \arrow[d, "\nu_*^{\overline{L}}"] & H^{k-1}(\mathbb{S}U) \arrow[r] \arrow[d, "\mathrm{id}"] & \cdots \\
\cdots \arrow[r] & H^{k-2}(\mathbb{S}U) \arrow[r] & \mathcal{H}^{k-1}(\mathbb{S}U, \pi) \arrow[r] & H^{k-1}(Y) \arrow[r, "-\pi^*"] & H^{k-1}(\mathbb{S}U) \arrow[r] & \cdots
\end{tikzcd}$$

Since the cohomologies of $\overline{L}$ and Y are isomorphic via $\nu_*^{\overline{L}}$ the Five Lemma shows that

$$\mathcal{H}^k\left(\mathbb{S}U, \overline{L}\right) \xrightarrow[\cong]{\nu_*^{\overline{L}}} \mathcal{H}^{k-1}(\mathbb{S}U, \pi) \cong H_c^{k-1}(U) \cong H_c^k(W)$$

as required. □

For future reference we point out that from the proof of this lemma and that of Lemma 1.5 the map

$$\Phi_{\overline{L}} : \mathcal{C}_c^\infty\left(W; \Lambda^k\right) \to \mathcal{C}_\pm^\infty\left(\overline{L}; \Lambda^k\right) \oplus \mathcal{C}^\infty\left(\mathbb{S}U; \Lambda^{k-2}\right) \tag{1.29}$$

defined by $\Phi_{\overline{L}}(\omega) = \left(i_{\overline{L}}^*\omega, (-1)^{k-1}\nu_*^{\overline{L}} R_*^W \omega\right)$, where i_L is the inclusion of L into W and $\nu_*^{\overline{L}}$ is the push-forward from $\mathbb{S}W$ to $\mathbb{S}U$, induces an isomorphism in cohomology between $H_c^k(W)$ and $\mathcal{H}^{k-1}\left(\mathbb{S}U, \overline{L}\right)$.

1.5. **Bundle over manifold with boundary.** A more general case, which arises in the index formula for the scattering algebra, corresponds to a vector bundle W over a compact manifold with boundary X. Let $\overline{W}$ be the radial compactification of W. Its boundary consists of two hypersurfaces, $\mathbb{S}W$, the part 'at infinity' and $\overline{W}_{\partial X}$, the part over the boundary. Set

$$\begin{aligned} &\mathcal{C}_1 = \mathcal{C}^\infty(X; \Lambda^*) \text{ and} \\ &\mathcal{C}_2 = \left\{(\alpha, \beta); \alpha \in \mathcal{C}^\infty(\mathbb{S}W; \Lambda^*),\ \beta \in \mathcal{C}^\infty(\overline{W}_{\partial X}; \Lambda^*) \text{ and } \iota_\partial^*\alpha = \iota_\partial^*\beta\right\}, \end{aligned} \tag{1.30}$$

the pairs of forms on the two hypersurfaces with common restriction to the corner $\mathbb{S}W_{\partial X}$. This gives a complex as in (1.3) with $\mathcal{C}^k = \mathcal{C}_1^k \oplus \mathcal{C}_2^{k-1}$ and differential

$$D = \begin{pmatrix} d & 0 \\ \phi & -d \end{pmatrix}, \text{ where } \phi = \begin{pmatrix} -\pi^* \\ -\pi^* i^* \end{pmatrix} \tag{1.31}$$

Lemma 1.7. *The cohomology of the complex* (1.30) *with differential* (1.31) *reduces to the cohomology with compact support in* $W_{X\setminus\partial X}$.

Proof. The cohomology of $\overline{W}_{\partial X}$ is canonically isomorphic to the cohomology of ∂X under the pull-back map. Thus a closed form on $\overline{W}_{\partial X}$ is the sum of a form on ∂X pulled-back to $\overline{W}_{\partial X}$ and an exact form on $\overline{W}_{\partial X}$. We point out that the same is true for a form β on $\overline{W}_{\partial X}$ if $d\beta$ is a form pulled-back from ∂X (this follows from the Hodge decomposition of forms and the previous statement or from the proof of [4, Cor. 4.1.2.2]).

Thus from $D(a, (\alpha, \beta)) = 0$ it follows that $\beta = \pi^*\beta' + d\beta''$ and hence

$$da = 0, \quad -\pi^* a = d\alpha, \quad -i^* a = d\beta'. \tag{1.32}$$

Choose a product neighborhood of the boundary $\mathcal{U} \cong [0,1]_x \times \partial X$ and a corresponding decomposition of $\mathbb{S}W\big|_{\mathcal{U}} \cong [0,1]_x \times V$ with $V = \mathbb{S}W\big|_{\partial X}$. In this neighborhood

$$a = a_t(x) + a_n(x) \wedge dx, \quad a_t \in \mathcal{C}^\infty(\partial X; \Lambda^k),\ a_n \in \mathcal{C}^\infty(\partial X; \Lambda^{k-1}), \tag{1.33}$$

$$\alpha = \alpha_t(x) + \alpha_n(x) \wedge dx, \quad \alpha_t \in \mathcal{C}^\infty(V; \Lambda^{k-1}),\ \alpha_n \in \mathcal{C}^\infty(V; \Lambda^{k-2}), \tag{1.34}$$

and (1.32) becomes

$$d_{\partial X} a_t = 0, \quad (-1)^k \partial_x a_t + d_{\partial X} a_n = 0, \quad d_V \alpha_t = -\pi^* a_t,$$
$$(-1)^k \partial_x \alpha_t + d_V \alpha_n = -\pi^* a_n, \quad -a_t(0) = d\beta'.$$

Choose a smooth function $\rho \in \mathcal{C}^\infty(X)$ that is identically equal to one if $x < 1/2$ and identically equal to zero if $x > 3/4$ and define $T \in \mathcal{C}^{k-1}$ by

$$T = \rho(x)\left((-1)^{k+1} \int_0^x a_n(s)\ ds - \beta', \left((-1)^{k-1} \int_0^x \alpha_n(s)\ ds + \iota_\partial^* \beta'', -\beta'' \right) \right).$$

Then, for some $\omega \in \mathcal{C}^k$,

$$DT = \rho(x)(a, (\alpha, \beta)) + \rho'(x)\omega$$

so that $(a, (\alpha, \beta)) - DT = (\widetilde{a}, (\widetilde{\alpha}, 0))$ with $\widetilde{a}$ and $\widetilde{\alpha}$ forms supported in X°. Thus the complex retracts to the subcomplex of forms supported in X°, and from Lemma 1.5 this complex computes the cohomology with compact support of W over X°. □

1.6. **Bundle with line subbundle over the boundary.** The representations of relative cohomology that will be used for the index of zero operators and for operators in Boutet de Monvel's transmission calculus are closely related. Consider a compact manifold with boundary, X, a real vector bundle W over X which over the boundary has a trivial line subbundle L, and denote the quotient bundle over the boundary by $U = W_{\partial X}/L$. The compactly supported cohomology of W will be represented as in §1.3, that of its restriction to the boundary either as in §1.4 for the zero calculus or §1.3 for the transmission calculus. An appropriate version of the inclusion map then gives the compactly supported cohomology of the interior much as in §1.2.

Notice that $\mathbb{S}W_{\partial X} \setminus \{L^+, L^-\}$ fibers over $\mathbb{S}U$ and can be identified with $\mathbb{S}U \times \mathbb{R}$ (since L is trivial), thus there is a push-forward map $\nu_*^{\mathbb{S}W} : \mathcal{C}^\infty\left(\mathbb{S}W, \Lambda^k\right) \to \mathcal{C}^\infty\left(\mathbb{S}U, \Lambda^{k-1}\right)$ which however does not commute with d.

Lemma 1.8. *If $\alpha \in \mathcal{C}^\infty\left(\mathbb{S}W, \Lambda^k\right)$ then*

$$\nu_*^{\mathbb{S}W} d_{\mathbb{S}W} \alpha = d_{\mathbb{S}U} \nu_*^{\mathbb{S}W} \alpha + (-1)^k (\pi^* i_{L^+} \alpha - \pi^* i_{L^-} \alpha). \tag{1.35}$$

Thus if $i_{L^+}\alpha = i_{L^-}\alpha$ then $\nu_L^* d\alpha = d\nu_L^* \alpha$.

Proof. Introduce polar coordinates around $L^\pm$ in $\mathbb{S}W_{\partial X}$ (i.e., blow them up) to get a map

$$\mathbb{S}U \times \overline{L} \xrightarrow{\beta} \mathbb{S}W_{\partial X}. \tag{1.36}$$

The pre-image of $L^\pm$ will still be denoted $L^\pm$. The push-forward is given by $a \mapsto \nu_*^{\overline{L}} \beta^* a$ and there are no integrability issues since $\mathbb{S}W_{\partial X}$ is compact.

In local coordinates, for a a form of degree k,

$$\beta^* a = a'(s) + a''(s) \wedge ds, \text{ with } a', a'' \in \mathcal{C}^\infty(\mathbb{S}U, \Lambda^*)$$
$$\implies d\beta^* a = d_{\mathbb{S}U} a'(s) + \left(d_{\mathbb{S}U} a''(s) + (-1)^k \partial_s a'(s) \right) \wedge ds.$$

Hence $\nu_*^{\overline{L}}\beta^* a = \int_{\overline{\mathbb{R}}} a''(s)\ ds$ and

$$\nu_*^{\overline{L}}\beta^*(d_{\mathbb{S}W}a) = d_{\mathbb{S}U}\nu_*^{L}\beta^* a + (-1)^k \int_{\overline{\mathbb{R}}} \partial_s a'(s)\ ds$$

giving (1.35). □

Introducing the complexes and chain maps

$$\mathcal{Z}_1^k = \mathcal{C}^\infty(X;\Lambda^k),\ \ \mathcal{Z}_3^k = \mathcal{C}^\infty(\mathbb{S}U;\Lambda^k)$$
$$\mathcal{Z}_2^k = \left\{(\alpha,\gamma)\in\mathcal{C}^\infty(\mathbb{S}W;\Lambda^k)\oplus\mathcal{C}^\infty(\overline{L};\Lambda^k); i_\pm^*\alpha = i_\pm^*\gamma\right\}$$
$$\phi_1 = \begin{pmatrix} -\pi_{\mathbb{S}W}^* \\ -\pi_{\overline{L}}^*\iota_\partial^* \end{pmatrix},\ \ \phi_2 = \left(-\nu_*^{\mathbb{S}W}, \pi_{\mathbb{S}U}^*\nu_*^{\overline{L}}\right),$$

we define the total chain space $\mathcal{Z}^*$ by

$$\mathcal{Z}^k = \mathcal{Z}_1^k\oplus\mathcal{Z}_2^{k-1}\oplus\mathcal{Z}_3^{k-3},\quad D_{\mathcal{Z}} = \begin{pmatrix} d & & \\ \phi_1 & -d & \\ & \phi_2 & d \end{pmatrix} \tag{1.37}$$

where $i_\pm$ are the inclusion (or attaching) maps for the two boundary manifolds (each canonically diffeomorphic to ∂X) $L_\pm$ of $\overline{L}$, either to $\mathbb{S}W$ or $\overline{L}$, and $\pi_{\mathbb{S}W}:\mathbb{S}W\longrightarrow X$, $\pi_{\mathbb{S}U}:\mathbb{S}U\longrightarrow\partial X$, and $\pi_{\overline{L}}:\overline{L}\longrightarrow\partial X$ denote the various bundle projections.

As in Lemma 1.8, neither the push-forward for α nor γ commute with d, however the consistency condition $i_\pm^*\alpha = i_\pm^*\gamma$ yields $d\phi_2 = \phi_2 d$; together with $d\phi_1 = \phi_1 d$ this ensures that $D_{\mathcal{Z}}^2 = 0$.

With the same notation set

$$\mathcal{C}_\pm^\infty(\mathbb{S}W;\Lambda^*) = \{\alpha\in\mathcal{C}^\infty(\mathbb{S}W;\Lambda^*) : i_{L^+}^*\alpha = i_{L^-}^*\alpha,\ i_{L^+}^*d\alpha = i_{L^-}^*d\alpha\}$$
$$\mathcal{T}_1^k = \mathcal{C}^\infty(X;\Lambda^k),\quad \mathcal{T}_2^k = \mathcal{C}_\pm^\infty(\mathbb{S}W;\Lambda^k)\oplus\mathcal{C}^\infty(\partial X;\Lambda^{k-1}),\quad \mathcal{T}_3^k = \mathcal{C}^\infty(\mathbb{S}U;\Lambda^k)$$
$$\Phi_1 = \begin{pmatrix} -\pi_{\mathbb{S}W}^* \\ i_{\partial X}^* \end{pmatrix}, \Phi_2 = \left(\nu_*^{\mathbb{S}W}, \pi_{\mathbb{S}U}^*\right)$$

and define the total chain space $\mathcal{T}^*$ by

$$\mathcal{T}^k = \mathcal{T}_1^k\oplus\mathcal{T}_2^{k-1}\oplus\mathcal{T}_2^{k-3},\quad D_{\mathcal{T}} = \begin{pmatrix} d & & \\ \Phi_1 & -d & \\ & \Phi_2 & d \end{pmatrix}. \tag{1.38}$$

Note that $D_{\mathcal{T}}^2 = 0$ because $i_{L^+}^*\alpha = i_{L^-}^*\alpha$ guarantees that $d\nu_*^L = \nu_*^L d$.

The point of these rather involved constructions of the cohomology with compact supports is that the Chern character derived from the symbolic data (symbol and normal operator) of a fully elliptic zero operator has a natural representative in the chain space $\mathcal{Z}_*$, while the Chern character constructed from the symbols (interior and boundary) of a fully elliptic operator in the transmission calculus has a natural representative in the chain space $\mathcal{T}_*$.

Lemma 1.9. *The cohomology of the complexes* (1.37) *and* (1.38) *are isomorphic to the compactly supported cohomology of W restricted to the interior of X,*

$$\mathcal{H}^k(\mathcal{T}_*;D_T) = \mathcal{H}^k(\mathcal{Z}_*;D_Z) = H_c^k(W\big|_{X\setminus\partial X}). \tag{1.39}$$

Proof. The map

$$\mathcal{C}_\pm^\infty\left(\overline{L};\Lambda^k\right)\oplus\mathcal{C}^\infty\left(\mathbb{S}U;\Lambda^{k-2}\right)\ni(\gamma,\beta)\longmapsto\left(0,0,\gamma-\pi_{\overline{L}}^* i_{L^+}^*\gamma,\beta\right)\in\mathcal{C}_k\left(\mathcal{Z}_*\right)$$

fits into the short exact sequence of complexes

$$0 \to \mathcal{C}_* \left(\mathbb{S}U, \overline{L}\right) \to \mathcal{Z}_* \to \mathcal{C}_* \left(\mathbb{S}W, \pi\right) \to 0.$$

This in turn induces the long exact sequence in cohomology in the top row of

$$\begin{tikzcd}
\ldots \to \mathcal{H}^k(\mathbb{S}W,\pi) \arrow[r, "J"] & \mathcal{H}^k\left(\mathbb{S}U,\overline{L}\right) \arrow[r] & \mathcal{H}^{k+1}(\mathcal{Z}_*; D_Z) \arrow[r] & \mathcal{H}^{k+1}(\mathbb{S}W,\pi) \to \ldots \\
\ldots \to H_c^k(W) \arrow[u, "\Phi"] \arrow[r, "i_*"] & H_c^k\left(W\big|_{\partial X}\right) \arrow[u, "\Phi_{\overline{L}}"] \arrow[r, "\delta"] & H_c^{k+1}\left(W, W\big|_{\partial X}\right) \arrow[u, "\widetilde{\Phi}"] \arrow[r] & H_c^{k+1}(W) \arrow[u, "\Phi"] \to \ldots
\end{tikzcd}$$

where the connecting map J is induced by

$$\begin{aligned}
\mathcal{C}^\infty(X;\Lambda^k) \oplus \mathcal{C}^\infty(\mathbb{S}W;\Lambda^{k-1}) &\longrightarrow \mathcal{C}^\infty(\overline{L};\Lambda^k) \oplus \mathcal{C}^\infty(\mathbb{S}U;\Lambda^{k-2}) \\
(a,\alpha) &\longmapsto (\pi_{\overline{L}}^* i_\partial^* a, \nu_*^L i_\partial^* \alpha)
\end{aligned}$$

Φ and $\Phi_{\overline{L}}$ are the isomorphisms defined in (1.26) and (1.29) respectively, and $\widetilde{\Phi}$ is the restriction of Φ to the subcomplex of forms that vanish at $W\big|_{\partial X}$.

The left and right squares above clearly commute, so we only check the commutativity of the middle square. Let u be a k-form on $W\big|_{\partial X}$ with compact support, the map δ is induced by taking any extension of u into W, say $e(u)$, and taking its exterior derivative. It is convenient to find $\gamma(u)$ such that $(\Phi e(u), \gamma(u), 0)$ is in $\mathcal{Z}_*$. To this end choose a trivialization of $\overline{L}$, denote by t the fibre variable along $\overline{L}$, and note that

$$\gamma(u)(t) = (-1)^{k-1} \int_0^t i_{\overline{L}}^* u$$

is as required and satisfies $d\gamma(u) = i_{\overline{L}}^* u - \pi_{\overline{L}}^* i_0^* u$ (since u is closed), and $\nu_*^{\overline{L}} \gamma(u) = 0$ (since $\mathrm{i}_{\partial_t}\gamma(u) = 0$).

Thus

$$\begin{aligned}
\widetilde{\Phi}(\delta(u)) &= (\Phi de(u), 0, 0) = (D_{\mathbb{S}W}\Phi e(u), 0, 0) \\
&= D(\Phi e(u), \gamma(u), 0) + \left(0, 0, i_{\overline{L}}^* u, (-1)^{k-1} \nu_*^{\overline{L}} R_*^W u\right) \\
&= D(\Phi e(u), \gamma(u), 0) + (0, 0, \Phi_{\overline{L}} u),
\end{aligned}$$

which shows that the induced maps in cohomology commute. It then follows from the Five Lemma that the map $\widetilde{\Phi}$ is an isomorphism, i.e., $\mathcal{H}^k(\mathcal{Z}_*; D_Z) \cong H_c^k\left(W, W\big|_{\partial X}\right)$.

To see that the complex $\mathcal{C}_k(\mathcal{T}_*; D_T)$ represents $H_c^k\left(W, W\big|_{\partial X}\right)$ consider first the complex

$$\mathcal{C}^\infty(X;\Lambda^*) \oplus \left(\mathcal{C}^\infty(\mathbb{S}W;\Lambda^*) \oplus \mathcal{C}^\infty(\partial X;\Lambda^*)\right) \oplus \mathcal{C}^\infty\left(\mathbb{S}W\big|_{\partial X};\Lambda^*\right)$$

$$\text{with differential} \begin{pmatrix} d & & & \\ -\pi^* & -d & & \\ i^* & & -d & \\ & i^* & \pi^* & d \end{pmatrix}$$

which in view of §1.2 and §1.3 represents $H_c^k\left(W, W\big|_{\partial X}\right)$ and then note that the complex $\mathcal{C}_k(\mathcal{T}_*; D_T)$ is obtained from this complex by applying the push-forward along $\overline{L}$ which was shown in Lemma 1.6 to be an isomorphism. □

2. Pseudodifferential operators and relative K-theory

In the standard case of Atiyah and Singer, the index of a vertical family of Fredholm operators, A, acting on a superbundle $\mathbb{E} = E^+ \oplus E^-$ on the fibers of a fibration of closed manifolds $M \xrightarrow{\phi} B$ is naturally thought of as an element of the topological K-theory group of B, e.g.,

$$[\ker A] - [\mathrm{coker} A] \in K(B)$$

when $\ker A$ (and hence $\mathrm{coker} A$) is a bundle over B. On the other hand A itself, via its symbol

$$\sigma(A) \in \mathcal{C}^\infty(\mathbb{S}^* M/B; \hom \mathbb{E})$$

and the clutching construction, defines an element of $K_{\mathrm{c}}(T^*M/B)$ and the index factors through this map

$$\begin{tikzcd} \Psi^*(M/B;\mathbb{E}) \arrow[rr, "\mathrm{Ind}"] \arrow[dr, "{[a]}"'] & & K(B)\,. \\ & K_{\mathrm{c}}(T^*M/B) \arrow[ur, dashed, "\mathrm{Ind_a}"'] & \end{tikzcd}$$

One way to see this factorization is to start with a family A as above, say made up of pseudodifferentical operators of order zero, and eliminate properties that the index does not see. That is, let $\mathcal{K}(M/B)$ be the set of equivalence classes of vertical pseudodifferential operators of order zero acting on superbundles $\mathbb{E}$ over M where two operators are considered equivalent if they can be connected by a finite sequence of relations:

i) $A \in \Psi^0(M/B;\mathbb{E}) \sim B \in \Psi^0(M/B;\mathbb{F})$ if there is a (graded) bundle isomorphism $\Phi : \mathbb{E} \to \mathbb{F}$ over M such that $B = \Phi^{-1} A \Phi$,

ii) $A \in \Psi^0(M/B;\mathbb{E}) \sim \widetilde{A} \in \Psi^0(M/B;\mathbb{E})$ if A and $\widetilde{A}$ are homotopic within elliptic operators,

iii) $A \in \Psi^0(M/B;\mathbb{E}) \sim A \oplus \mathrm{Id} \in \Psi^0\left(M/B;\mathbb{E} \oplus \mathbb{C}^{n|n}\right)$ where $\mathbb{C}^{n|n}$ is the trivial superbundle whose $\mathbb{Z}/2$ grading components are both $\mathbb{C}^n$.

The resulting equivalence classes form a group, $\mathcal{K}(M/B)$ which can be thought of as 'smooth K-theory' and in this case is well-known to coincide with the topological K-theory group $K_{\mathrm{c}}(T^*M/B)$. Indeed, the equivalence class of an operator only depends on its principal symbol

$$A \in \Psi^0(M/B;\mathbb{E}) \implies \sigma(A) \in \mathcal{C}^\infty(S^*M/B; \hom \mathbb{E})$$

in terms of which (i)-(iii) give a standard 'relative' definition of $K_{\mathrm{c}}(T^*M/B)$.

Alternately, we can think of pseudodifferential operators of order 0 as bounded operators acting on $L^2(M/B)$ (defined, e.g., using a Riemannian metric). These form a $*$-algebra and the closure is a C^*-algebra, $\mathfrak{A}$, containing the compact operators, $\mathfrak{K}$. The Fredholm operators are the invertibles in $\mathfrak{A}/\mathfrak{K}$, and the smooth K-theory group described above is closely related to the odd C^* K-theory group of this quotient, $K^1_{C^*}(\mathfrak{A}/\mathfrak{K})$. Indeed, the principal symbol extends to a continuous map on $\mathfrak{A}$,

$$A \in \mathfrak{A} \implies \sigma(A) \in \mathcal{C}^0(S^*M/B;\mathbb{C}),$$

which descends to the quotient $\mathfrak{A}/\mathfrak{K}$ (i.e., vanishes on $\mathfrak{K}$) and allows us to identify the odd K-theory group with the stable homotopy classes of invertible maps $S^*M/B \to \mathbb{C}$.

The most obvious difference between the smooth K-theory group and the C^* K-theory group – that the former is built up from smooth functions while the latter from continuous functions – disappears in the quotient. A more significant difference comes from the way bundle coefficients are handled. Stabilization allows us to replace the elliptic elements in $\Psi^0(M/B)$ with $\varinjlim \mathrm{GL}_N\left(\Psi^0(M/B)\right)$, the direct limit of the groups of invertible square matrices of arbitrary size and entries in $\Psi^0(M/B)$. Note that an operator $A \in \Psi^0(M/B;\mathbb{E})$ acting on a superbundle $\mathbb{E}$ defines an element of the stabilization of $\Psi^0(M/B)$ by choosing a vector bundle F such that $\mathbb{C}^j \cong \mathbb{E} \oplus F$ (for some j) since then $\widetilde{A} = A \oplus \mathrm{Id}_F \in \mathrm{GL}_j\left(\Psi^0(M/B)\right)$; a different choice of F defines the same element in $K^1_{C^*}(M/B)$. However if Φ is as in (i) above, it is possible that $\Phi^{-1}A\Phi$ and A will define distinct elements of $K^1_{C^*}(M/B)$, and so the difference between the smooth K-theory groups and the C^* K-theory groups is essentially that in the former we quotient out by (i) above. This is further pursued in [1, §2.3].

In the present paper we allow the fibers of the fibration $M \xrightarrow{\phi} B$ to have boundary and consider three different calculi of operators on a manifold with boundary, namely the scattering calculus, the zero calculus, and the transmission calculus. In each case we work with the smooth K-theory group as in the previous paragraph (denoted by $\mathcal{K}_{sc}(M/B)$, $\mathcal{K}_0(M/B)$, and $\mathcal{K}_{\mathrm{tm}}(M/B)$ respectively) and, for the purposes of index theory, these groups contain all of the relevant information. These groups have been shown to be isomorphic to the topological K-theory group, $K_{\mathrm{c}}(T^*M^\circ/B)$, in [17] [21] (scattering calculus), [1] (0-calculus), and [5] (transmission calculus). We briefly review what this entails.

Scattering calculus. A scattering operator $A \in \Psi^0_{sc}(M/B;\mathbb{E})$ is determined up to a compact operator by its image under two homomorphisms: the principal symbol $\sigma(A) \in \mathcal{C}^\infty(\mathbb{S}^*M/B;\pi^*\hom\mathbb{E})$ which is an homomorphism between the lifts of E^+ and E^- to $\mathbb{S}^*M/B$, and the boundary symbol $b \in \mathcal{C}^\infty(\overline{T^*M/B}_{\partial M};\pi^*\hom(\mathbb{E}))$ which is an homomorphism between the lifts of E^+ and E^- to the radial compactification of T^*M/B over the boundary. These symbols are equal on the common boundary of $\mathbb{S}^*M/B$ and $\overline{T^*M/B}_{\partial M}$, and so can be thought of as jointly representing a section of $\hom(\mathbb{E})$ lifted to the whole boundary of the compact manifold with corners $\overline{T^*M/B}$. An operator is Fredholm on L^2 precisely when both of these symbols are invertible, we call such an operator 'fully elliptic'.

A fully elliptic operator A can be deformed by homotopy within such operators operators until b, and a in a neighborhood of the boundary, are equal to a fixed bundle isomorphism. This isomorphism can then be used to change the bundles so that b is the identity and a is the identity near the boundary. This leads to an (arbitrary) invertible map into $\hom(\mathbb{E})$ that is the identity in an neighborhood of the boundary and this is precisely the information that defines a relative K-theory class, hence

$$\mathcal{K}_{sc}(M/B) = K_{\mathrm{c}}(T^*M^\circ/B). \tag{2.1}$$

The fully elliptic scattering operators of order zero with interior symbol equal to the identity can be reduced by homotopy to perturbations of the identity by scattering operators of order $-\infty$. The smooth K-theory of these perturbations of the identity is denoted $\mathcal{K}_{sc,-\infty}(\partial M/B)$ and is readily seen to be equal to the

topological K-theory of the boundary,

$$\mathcal{K}_{-\infty,sc}(M/B) = K_{\mathrm{c}}(T^*\partial M/B)\,. \tag{2.2}$$

Zero calculus. The analogues of (2.1) and (2.2) also hold for the zero calculus, as shown in [1]. However where the scattering calculus is 'asymptotically commutative' as evinced in the boundary symbol $b \in \mathcal{C}^\infty(\overline{T^*M/B}_{\partial M}; \pi^* \hom(\mathbb{E}))$, the zero calculus is asymptotically commutative only in the directions tangent to the boundary and is non-commutative in the direction normal to the boundary. Thus, instead of a boundary symbol, the boundary behavior of a zero operator is captured by a family of operators on a one-dimensional space, $\mathcal{I}$, essentially the compactified normal bundle to the boundary. This family, the reduced normal operator

$$\mathcal{N}(A) \in \mathcal{C}^\infty\left(S^*\partial M/B; \Psi^0_{b,c}(\mathcal{I};\mathbb{E})\right),$$

takes values in the b,c calculus (b at one end of the interval, c at the other), and together with the interior symbol, determines the smooth K-theory class of an operator $A \in \Psi^0_0(M/B;\mathbb{E})$. A description of the reduced normal operator is included in Appendix A. As an element of the b,c calculus, $\mathcal{N}(A)$ has three model operators: its principal symbol, an indicial family at the b-end, and an indicial family at the cusp end. One can think of the smooth K-theory as equivalence classes of zero pseudodifferential operators or alternately as equivalence classes of invertible pairs

$$\begin{gathered}(\sigma,\mathcal{N}) \in \mathcal{C}^\infty(S^*M/B; \hom\mathbb{E}) \oplus \mathcal{C}^\infty\left(S^*\partial M/B; \Psi^0_{b,c}(\mathcal{I};\mathbb{E})\right)\\ \text{s.t. } I_b(\mathcal{N}(y,\eta)) = I_b(\mathcal{N}(y,\eta'))\,,\\ I_c(\mathcal{N}(y,\eta))(\xi) = \sigma\left(0,y,\frac{\xi}{\langle\xi\rangle},\frac{\eta}{\langle\xi\rangle}\right),\\ \sigma(\mathcal{N}(y,\eta))(\omega) = \sigma(0,y,\omega,0)\,.\end{gathered} \tag{2.3}$$

The isomorphism

$$\mathcal{K}_{-\infty,0}(M/B) \cong K_{\mathrm{c}}(T^*\partial M/B)\,,$$

between the group of stable equivalence classes of invertible reduced normal families of perturbations of the identity and a standard presentation of $K_{\mathrm{c}}(T^*\partial M/B)$ comes from the contractibility of the underlying semigroup of invertible operators on the interval [1]. Namely, this allows the reduced normal family to be connected (after stabilization) to the identity through a curve of maps $A(t)$ from $S^*\partial M/B$ into the invertible b-operators of the form $\mathrm{Id} + A$, A of order $-\infty$ and non-trivial only at the one end of the interval. The b-indicial family of $A(t)$ determines an invertible map from $\overline{T^*\partial M/B}$ into $\hom\mathbb{E}$ equal to the identity at infinity and hence an element of $K_{\mathrm{c}}(T^*\partial M/B)$.

The contractibility of the group of invertible group of smooth perturbations of the identity within the cusp calculus [20] is used to show the isomorphism

$$\mathcal{K}_0(M/B) \cong K_{\mathrm{c}}(T^*M^\circ/B)$$

between the group of stable equivalence classes of invertible pairs (2.3) and the standard representation of $K_{\mathrm{c}}(T^*M^\circ/B)$. Indeed, one can identify E^+ and E^- near the boundary, and, after a homotopy and a smooth perturbation, quantize by a zero operator whose *full* b-indicial family is the identity, and then use the contractibility to take the reduced normal operator to the identity. The principal

symbol of the resulting operator is equal to the identity near the boundary and classically defines an element of $K_{\mathrm{c}}(T^*M^\circ/B)$.

Transmission calculus. A smooth K-theory class in the Boutet de Monvel calculus is an equivalence class of operators of the form

$$\mathcal{A} = \begin{pmatrix} \gamma^+ A + B & K \\ & \\ T & Q \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty(X; E^+) \\ \oplus \\ \mathcal{C}^\infty(\partial X; F^+) \end{matrix} \to \begin{matrix} \mathcal{C}^\infty(X; E^-) \\ \oplus \\ \mathcal{C}^\infty(\partial X; F^-) \end{matrix} \tag{2.4}$$

acting on the superbundles $\mathbb{E}$ over X and $\mathbb{F}$ over ∂X. The principal symbol of A is required to satisfy the *transmission condition* at the boundary, and in particular this forces the order of A to be an integer. We can assume without loss of generality that the order of A and its 'type' are both zero (see [5], [7]).

The boundary behavior of A is modeled by a family of Wiener-Hopf operators parametrized by the cosphere bundle over the boundary. For A as above we denote its boundary symbol by

$$N(A)(y,\eta) = \begin{pmatrix} h^+ p + b & k \\ & \\ t & q \end{pmatrix} : \begin{matrix} \mathcal{C}^\infty\left(X; H^+ \otimes E_y^+\right) \\ \oplus \\ \mathcal{C}^\infty\left(\partial X; F_y^+\right) \end{matrix} \to \begin{matrix} \mathcal{C}^\infty\left(X; H^+ \otimes E_y^-\right) \\ \oplus \\ \mathcal{C}^\infty\left(\partial X; F_y^-\right) \end{matrix} \tag{2.5}$$

where h^+ is the projection from the space of functions $\mathcal{C}^\infty(\mathbb{R};\mathbb{C})$ that have a regular pole at infinity to the subspace H^+ of those that vanish at infinity and can be continued analytically to the lower half-plane.

Boutet de Monvel showed that any operator A whose principal symbol and boundary symbol are both invertible is homotopic through such operators to one of the form

$$\begin{pmatrix} \gamma^+ \widetilde{A} & 0 \\ 0 & \widetilde{Q} \end{pmatrix}$$

where $\widetilde{A}$ is equal to the identity near the boundary. Thus $\mathcal{K}_{\mathrm{tm}}(M/B)$ clearly surjects onto $K_{\mathrm{c}}(T^*M^\circ/B)$ which is all we will need. However we point out that, using the description of the C^*-algebra K-theory from, e.g., [12], [13], and the comparison with smooth K-theory in [1, §2.3], it is possible to show that $\mathcal{K}_{\mathrm{tm}}(M/B)$ is actually equal to $K_{\mathrm{c}}(T^*M^\circ/B)$.

3. Chern character and the families index theorem

The Chern character is a homomorphism from $\mathrm{Ch} : K(X) \longrightarrow H^{\mathrm{even}}(X)$ which gives an isomorphism after tensoring with $\mathbb{Q}$. Chern-Weil theory gives a direct representation of the Chern character in deRham cohomology for a superbundle $\mathbb{E}$. If ∇ is a graded connection on $\mathbb{E}$, i.e. a pair of connections $\nabla^\pm$ on $E^\pm$ with curvatures $(\nabla^\pm)^2 = -2\pi i \omega_\pm$ then

$$\mathrm{Ch}(\mathbb{E}) = e^{\omega_+} - e^{\omega_-}. \tag{3.1}$$

Different choices of connection are homotopic and give cohomologous closed forms.

In the case of a compact manifold with boundary X, the K-theory with compact support in the interior, denoted here $K_{\mathrm{c}}(X \setminus \partial X)$, is represented by superbundles $\mathbb{E}$ where $E^\pm = \mathbb{C}^N$ near the boundary. Then the same formula, (3.1), gives the relative Chern character $\mathrm{Ch} : K_{\mathrm{c}}(X \setminus \partial X) \longrightarrow H_{\mathrm{c}}^{\mathrm{even}}(X \setminus \partial X)$ provided the connections are chosen to reduce to the trivial connection, d, near the boundary.

There are natural isomorphism $K_c(X \setminus \partial X) \longrightarrow \mathrm{K}(X, \partial X)$ and $H_c^{\mathrm{even}}(X \setminus \partial X) \longrightarrow H^{\mathrm{even}}(X, \partial X)$ with the corresponding relative objects. Chains for $K_c(X \setminus \partial X)$ are given by pairs $(\mathbb{E}, a)$ of a superbundle over X and an isomorphism $a : E^+ \longrightarrow E^-$ over ∂X. In [6] Fedosov gives an explicit formula for the Chern character, in cohomology with compact supports of the cotangent bundle, of the symbol of an elliptic operator acting between vector bundles. This can be modified to give the relative Chern character in this setting with values in the chain space discussed in Lemma 1.3

$$\mathrm{Ch}(\mathbb{E}, a) = (\mathrm{Ch}(\mathbb{E}), \widetilde{\mathrm{Ch}}(a))$$

$$\widetilde{\mathrm{Ch}}(a) = -\frac{1}{2\pi i} \int_0^1 \mathrm{tr}\left(a^{-1}(\nabla a) e^{w(t)}\right) dt, \tag{3.2}$$

$$\text{where } w(t) = (1-t)\omega_+ + t a^{-1}\omega_- a + \frac{1}{2\pi i} t(1-t)(a^{-1}\nabla a)^2.$$

Here the boundary term, $\widetilde{\mathrm{Ch}}(a)$, is a 'regularized' (or improper) form of the odd Chern character.

In the context of the index formula this also corresponds rather naturally to the relative cohomology as discussed above. Essentially by reinterpretation we find

Proposition 3.1 (Fedosov [6])**.** *If $\pi : U \longrightarrow X$ is a real vector bundle, $\mathbb{E} \longrightarrow X$ is a superbundle and $a \in \mathcal{C}^\infty(\mathbb{S}U; \pi^*(\mathrm{hom}(\mathbb{E}))$ is elliptic (i.e. invertible) then for any graded connection on $\mathbb{E}$, with curvatures $\Omega_\pm = -2\pi i \omega_\pm$ and induced connection ∇ on* $\mathrm{hom}(\mathbb{E})$, *the class*

$$\mathrm{Ch}(U, \mathbb{E}, a) = (\mathrm{Ch}(\mathbb{E}), \widetilde{\mathrm{Ch}}(a)) \in \mathcal{C}^\infty(X; \Lambda^{even}) \oplus \mathcal{C}^\infty(\mathbb{S}U; \Lambda^{odd}), \tag{3.3}$$

given by the formulæ(3.1) *and* (3.2)*, represents the relative Chern character*

$$K_c^0(U) \longrightarrow \mathcal{H}^{odd}(\mathbb{S}U, \pi) \simeq H_c^{even}(U). \tag{3.4}$$

Proof. That the pair $(\mathrm{Ch}(\mathbb{E}), \widetilde{\mathrm{Ch}}(a))$ is D-closed and gives a well-defined class in the cohomology theory follows in essence as in standard Chern-Weil theory. We include such an argument for completeness and for subsequent generalization.

Set $\theta = a^{-1}\nabla a$ and note that the connection $\widetilde{\nabla} = \nabla + [t\theta, \cdot]$ has curvature

$$\Omega(t) = -2\pi i \omega_+ + \left(t\nabla\theta + t^2\theta^2\right) = -2\pi i \omega(t)$$

It follows that $\widetilde{\nabla} e^{\omega(t)} = 0$ and hence

$$\nabla e^{\omega(t)} = \widetilde{\nabla} e^{\omega(t)} - t\left[\theta, e^{\omega(t)}\right] = t\left[e^{\omega(t)}, \theta\right].$$

Thus

$$\begin{aligned} d\,\widetilde{\mathrm{Ch}}(a) &= -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\,\nabla\left(\theta e^{\omega(t)}\right)\, dt \\ &= -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left((\nabla\theta)\, e^{\omega(t)} - \theta\nabla e^{\omega(t)}\right)\, dt \\ &= -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left((\nabla\theta)\, e^{\omega(t)} - t\theta\left[e^{\omega(t)}, \theta\right]\right)\, dt \\ &= -\frac{1}{2\pi i}\int_0^1 \mathrm{tr}\left(\left(\nabla\theta + 2t\theta^2\right) e^{\omega(t)} - t\left[\theta e^{\omega(t)}, \theta\right]\right)\, dt \end{aligned} \tag{3.5}$$

and since the trace vanishes on graded commutators,

$$d\,\widetilde{\mathrm{Ch}}(a) = \int_0^1 \mathrm{tr}\left(\frac{d\omega(t)}{dt}e^{\omega(t)}\right)dt = \pi^*\,\mathrm{tr}(e^{\omega_-}) - \pi^*\,\mathrm{tr}(e^{\omega_+}) = -\pi^*\,\mathrm{Ch}(\mathbb{E}). \tag{3.6}$$

As is well-known (and explained in §1.1.2), this same formula shows independence of the connection and homotopy invariance. That this class actually represents the (appropriately normalized) Chern character follows from Fedosov's derivation in [6]. □

For an elliptic family of pseudodifferential operators $A \in \Psi^0(M/B;\mathbb{E})$ where $\psi : M \longrightarrow B$ is a fibration with typical fibre Z and $\mathbb{E}$ is a superbundle over M, the symbol $\sigma(A) \in \mathcal{C}^\infty(S^*(M/B);\pi^*\hom(\mathbb{E}))$ is invertible (by assumption) and the discussion above applies with $U = T^*(M/B)$, the fibre cotangent bundle. The index formula of Atiyah and Singer is then given as a composite

$$\begin{tikzcd} K_{\mathrm{c}}(T^*(M/B)) \arrow[rr, dashed, "\mathrm{ind}"] \arrow[d, "\mathrm{Ch}"] & & H^{\mathrm{even}}(B) \\ \mathcal{H}^{\mathrm{odd}}(S^*(M/B),\pi) \arrow[rr, "\wedge\,\mathrm{Td}(Z)"] & & \mathcal{H}^{\mathrm{odd}}(S^*(M/B),\pi) \arrow[u, "\int_{S^*Z}"'] \end{tikzcd} \tag{3.7}$$

So, for such a family of operators,

$$\mathrm{Ch}(\mathrm{ind}(A)) = \int_{S^*Z} \mathrm{Ch}(T^*(M/B),\mathbb{E},\sigma(A)) \wedge \mathrm{Td}(Z) \tag{3.8}$$

where $\mathrm{Td}(Z)$ is the Todd class of Z. That this is well-defined follows from (1.15).

3.1. **Scattering families index theorem.** Consider next a fibration of manifolds with boundary $M \xrightarrow{\psi} B$. As described in §2 the compactly supported K-theory of $W = T^*M^\circ/B$ can be represented by scattering operators. The K-theory class of a scattering operator is determined by its two symbol maps, its principal symbol and its boundary symbol. We now explain how to represent the Chern character of the corresponding K-theory class in terms of this data.

In fact given any manifold with boundary M and a bundle $W \to M$ any class in the compactly supported K-theory of $W\big|_{M\setminus\partial M}$ can be represented by a superbundle $\mathbb{E} \to M$ and two invertible maps

$$a \in \mathcal{C}^\infty(\mathbb{S}^*W;\pi^*\hom\mathbb{E}),\ \ b \in \mathcal{C}^\infty(\overline{W}_{\partial M};\pi^*\hom\mathbb{E}).$$

We recall, from §1.5, that one may compute the cohomology $H^*_c(W\big|_{M\setminus\partial M})$ via the complex

$$\mathcal{C}^\infty(M;\Lambda^k) \oplus \left\{(\alpha,\beta);\alpha \in \mathcal{C}^\infty(\mathbb{S}W;\Lambda^{k-1}),\ \beta \in \mathcal{C}^\infty(\overline{W}_{\partial M};\Lambda^{k-1}) \text{ and } \iota_\partial^*\alpha = \iota_\partial^*\beta\right\},$$
$$D = \begin{pmatrix} d & 0 \\ \phi & -d \end{pmatrix},\ \text{ where } \phi = \begin{pmatrix} -\pi^* \\ -\pi^* i^* \end{pmatrix}.$$

Then, choosing a graded connection on $\mathbb{E}$, the forms, again using (3.1) and (3.2),

$$\begin{aligned} \mathrm{Ch}(W,\mathbb{E},a,b) = &\left(\mathrm{Ch}\,(\mathbb{E})\,,\widetilde{\mathrm{Ch}}(a),\widetilde{\mathrm{Ch}}(b)\right) \\ &\in \mathcal{C}^\infty\left(X;\Lambda^{\mathrm{even}}\right) \oplus \mathcal{C}^\infty\left(\mathbb{S}W;\Lambda^{\mathrm{odd}}\right) \oplus \mathcal{C}^\infty\left(\overline{W}_{\partial M};\Lambda^{\mathrm{odd}}\right), \end{aligned} \tag{3.9}$$

represent the Chern character of the K-theory class associated to $(W,\mathbb{E},a,b)$.

Proposition 3.2. *For W, $\mathbb{E}$, a, and b as above,* $\mathrm{Ch}(W, \mathbb{E}, a, b)$ *is D-closed and its relative cohomology class coincides with the Chern character of the K-theory class defined by* $(W, \mathbb{E}, a, b)$.

Proof. From the definition of the differential in §1.5, D-closed means that

$$d_X \,\mathrm{Ch}\,(\mathbb{E}) = 0,\ d_{\mathbb{S}W}\,\widetilde{\mathrm{Ch}}\,(a) = -\pi^* \,\mathrm{Ch}\,(\mathbb{E}) \text{ and } d_{\overline{W}_{\partial X}}\,\widetilde{\mathrm{Ch}}\,(b) = -\pi^* i^* \,\mathrm{Ch}\,(\mathbb{E})\,.$$

Thus that the putative Chern character is D-closed and homotopy invariant follow just as in Lemma 3.1. It is also invariant under changes of $\mathbb{E}$ by stabilization and bundle isomorphism since this is true of the forms $\left(\mathrm{Ch}\,(\mathbb{E})\,, \widetilde{\mathrm{Ch}}(a), \widetilde{\mathrm{Ch}}(b)\right)$ themselves, and hence it only depends on the K-theory class.

As explained in [17] and reviewed in §2, there is a representative of the K-theory class with $b = \mathrm{Id}$ and $a = \mathrm{Id}$ near the boundary, and, since in this case (3.9) coincides with Fedosov's formula (3.3), we conclude that (3.9) is the usual Chern character map. □

The index formula follows similarly. Given a vertical family of fully elliptic scattering pseudodifferential operators acting on a superbundle $\mathbb{E} \to M$, it is possible to make a homotopy within fully elliptic scaterring operators until the symbols are bundle isomorphisms at and near the boundary. Thus a formula which is homotopy invariant and which coincides with the usual Atiyah-Singer index formula when the operators are trivial at the boundary must give the index.

Proposition 3.3. *The index in cohomology for a family of fully elliptic scattering pseudodifferential operators on the fibres of a fibration is given by the Atiyah-Singer formula essentially as in* (3.7), (3.8)*:*

$$\begin{aligned}\mathrm{ind}(A) &= \int \mathrm{Ch}(\mathbb{E}, \sigma(A), \beta(A)) \wedge \mathrm{Td}(Z) \\ &= \int_{{}^{\mathrm{sc}}\mathrm{S}^* Z} \widetilde{\mathrm{Ch}}\,(\sigma(A)) \wedge \mathrm{Td}(Z) + \int_{\overline{{}^{\mathrm{sc}}\mathrm{T}^*_{\partial Z}\, Z}} \widetilde{\mathrm{Ch}}\,(\beta(A)) \wedge \iota_\partial^* \,\mathrm{Td}(Z)\end{aligned} \tag{3.10}$$

where $\mathrm{Td}(Z) \in \mathcal{C}^\infty(M; \Lambda^{even})$ *is a deRham form representing the Todd class of the fibres of* $\psi : M \longrightarrow Y$.

Proof. Homotopy invariance follows from Proposition 3.2 and when the family of operators are trivial near the boundary this clearly coincides with the Atiyah-Singer families index formula. □

3.2. **Zero families index theorem.** The non-commutativity of the boundary symbol, in the normal direction, makes the derivation of a formula more challenging, so we first consider the simple case where only the boundary symbol appears.

Perturbations of the identity. Consider a fibration as in (1) where the typical fiber, X, is a manifold with boundary, denoted Y. The restriction of the fibrewise cotangent bundle to the boundary $T^*_{\partial M} M/B$ has a trivial line sub-bundle, the conormal bundle, with the quotient being $T^*\partial M/B$. Thus as explained in §1.3 the compactly supported cohomology of $U = T^*\partial M/B$ can be realized as the cohomology $\mathcal{H}^k\,(\mathbb{S}U, \pi)$ of the complex

$$\mathcal{C}^\infty(\partial M; \Lambda^*) \oplus \mathcal{C}^\infty(\mathbb{S}U; \Lambda^{*-1}),\ D = \begin{pmatrix} d & 0 \\ -\pi^* & -d \end{pmatrix}. \tag{3.11}$$

On the other hand, as explained in §2, the zero pseudodifferential operators on X can be used to realize the K-theory of Y. Indeed

$$K_{\mathrm{c}}\left(T^*\partial M/B\right)=\mathcal{K}_{0,-\infty}\left(M/B\right), \tag{3.12}$$

and, since the class of an operator $\mathrm{Id}+A$ (acting as an odd operator on sections of the superbundle $\mathbb{E}$) in the quotient on the right is determined by its reduced normal operator, any class in $K_{\mathrm{c}}\left(T^*\partial M/B\right)$ can be represented by a map $N\in\mathcal{C}^\infty\left(\mathbb{S}U,\Psi_{\mathrm{b}}^{-\infty}([0,1];\mathbb{E})\right)$.

A natural candidate for the Chern character, in view of the previous sections, would be to choose a graded connection on $\mathbb{E}$ and then consider $\widetilde{\mathrm{Ch}}(N)$. However, elements of $\Psi_{\mathrm{b}}^{-\infty}([0,1];\mathbb{E})$ are generally not trace-class and this corresponding expression is not well defined unless the trace is renormalized. In this setting, the 'b-trace'

$$\overline{\mathrm{Tr}}:\Psi_{\mathrm{b}}^{-\infty}([0,1];\mathbb{E})\longrightarrow\mathbb{C}$$

is an extension of the trace that however is not itself a trace. Indeed, instead of vanishing on commutators it satisfies a 'trace-defect formula', namely

$$\overline{\mathrm{Tr}}\left([A,B]\right)=\frac{1}{2\pi i}\int_{\mathbb{R}}\mathrm{Tr}\left(\frac{\partial a}{\partial\xi}b\right)\,d\xi, \tag{3.13}$$

where a and b are the indicial operators of A and B respectively (see [15, Chapter 4]). Renormalized traces are briefly discussed in Appendix B following [18].

Given a graded connection ∇ on $\mathbb{E}$ and a choice of boundary defining function x (on which the definition of the b-trace depends) consider the algebra of matrices with entries which are b-pseudodifferential operators on an interval. The b-trace will be used to define odd 'eta forms' on the semigroup of L^2-invertible order $-\infty$ perturbations of the identity. These are regularized versions of the odd Chern character given by (3.2). Taking into account the fact that $\overline{\mathrm{Tr}}$ is not a trace set

$$\begin{gathered}\eta_{\mathrm{b}}^{\mathrm{odd}}\left(N\right)=-\frac{1}{2\pi i}\int_0^1\left(1-t\right)\overline{\mathrm{Tr}}\left(N^{-1}\left(\nabla N\right)e^{w_N(t)}\right)+t\overline{\mathrm{Tr}}\left(\left(\nabla N\right)e^{w_N(t)}N^{-1}\right)\,dt\\ \text{where }w_N(t)=(1-t)\omega_++tN^{-1}\omega_-N+\frac{1}{2\pi i}t(1-t)(N^{-1}\nabla N)^2.\end{gathered} \tag{3.14}$$

The *even* Chern character of the indicial family of N is also needed here, since these are in essence loops into smoothing operators. The even Chern character therefore arises from $\widetilde{\mathrm{Ch}}$ by transgression

$$\mathrm{Ch}^{\mathrm{even}}\left(a\right)=\int_{\mathbb{R}}\mathrm{i}_{\partial_\xi}\,\widetilde{\mathrm{Ch}}_{Y\times\mathbb{R}}\left(a\right)\,d\xi. \tag{3.15}$$

Lemma 3.4. *If N is a family of a Fredholm zero operators of the form $\mathrm{Id}+A$ where $A\in\Psi_0^{-\infty}\left(M/B;\mathbb{E}\right)$ and $a=I_b\left(N\right)$ then*

$$d\eta_{\mathrm{b}}^{odd}\left(N\right)=-\pi^*\,\mathrm{Ch}\left(\mathbb{E}\right)+\nu_*^{\overline{L}}\,\widetilde{\mathrm{Ch}}\left(I_b(N)\right)=-\pi^*\,\mathrm{Ch}\left(\mathbb{E}\right)+\mathrm{Ch}^{even}\left(a\right) \tag{3.16}$$

where $\mathrm{Ch}\left(\mathbb{E}\right)$ *is given by* (3.2).

Proof. Set

$$\theta=N^{-1}\nabla N,\ \theta_a=a^{-1}\nabla a\text{ and }\theta_\xi=a^{-1}\frac{\partial a}{\partial\xi},$$

and recall, as in Proposition 3.1, that

$$\partial_t e^{w(t)} = -\frac{1}{2\pi i}\left(\nabla\left(\theta e^{w(t)}\right) + t\theta e^{w(t)}\theta + t\theta^2 e^{w(t)}\right).$$

Then

$$\begin{aligned}
d\eta\left(N\right) = d&\left[\frac{i}{2\pi}\int_0^1 \overline{\operatorname{Tr}}\left(\left(1-t\right)\theta e^{w_N(t)} + tN\theta e^{w_N(t)}N^{-1}\right)\,dt\right]\\
&= \frac{i}{2\pi}\int_0^1 \overline{\operatorname{Tr}}\left(\nabla\left(\left(1-t\right)\theta e^{w_N(t)} + tN\theta e^{w_N(t)}N^{-1}\right)\right)\,dt\\
&= \int_0^1 \overline{\operatorname{Tr}}\left(tN\partial_t e^{w_N(t)}N^{-1} + (1-t)\partial_t e^{w_N(t)}\right)\,dt\\
&+ \frac{i}{2\pi}\int_0^1 \overline{\operatorname{Tr}}\left(t\left(1-t\right)\left[N\left[\theta e^{w_N(t)},\theta\right],N^{-1}\right]\right)\,dt\\
&= \int_0^1 \partial_t\overline{\operatorname{Tr}}\left(tNe^{w_N(t)}N^{-1} + (1-t)e^{w_N(t)}\right)\,dt
\end{aligned}$$

$$+\int_0^1 \overline{\operatorname{Tr}}\left(\left[Ne^{w_N(t)},N^{-1}\right]\right)\,dt + \frac{i}{2\pi}\int_0^1 \overline{\operatorname{Tr}}\left(t\left(1-t\right)\left[N\left[\theta e^{w_N(t)},\theta\right],N^{-1}\right]\right)\,dt.$$

The integral of the t-derivative reduces to

$$\overline{\operatorname{Tr}}\left(Ne^{w_N(1)}N^{-1}\right) - \overline{\operatorname{Tr}}\left(e^{w_N(0)}\right) = -\pi^*\operatorname{Ch}\left(\mathbb{E}\right)$$

and the other terms can be evaluated using the trace-defect formula giving

$$\int_0^1 \frac{i}{2\pi}\int_{\mathbb{R}} \operatorname{tr}\left(\theta_\xi e^{w_a(t)}\right)\,d\xi\,dt + \frac{i}{2\pi}\int_0^1 \frac{i}{2\pi}t\left(1-t\right)\int_{\mathbb{R}}\operatorname{tr}\left(-\theta_\xi\left[\theta_a e^{w_a(t)},\theta_a\right]\right)\,d\xi\,dt.$$

Comparing this with

$$\begin{aligned}
\mathrm{i}_{\partial_\xi}\widetilde{\operatorname{Ch}}\left(a\right) &= \frac{i}{2\pi}\mathrm{i}_{\partial_\xi}\int_0^1 \operatorname{tr}\left(\theta_a e^{w_a(t)}\right)\,dt\\
&= \frac{i}{2\pi}\int_0^1 \operatorname{tr}\left(\theta_\xi e^{w_a(t)} - \frac{i}{2\pi}\theta_a e^{w_a(t)}\left(t\partial_\xi\left(\theta_a\right) - t\nabla\left(\theta_\xi\right) - t^2\left[\theta_a,\theta_\xi\right]\right)\right)\,dt\\
&= \frac{i}{2\pi}\int_0^1 \operatorname{tr}\left(\theta_\xi e^{w_a(t)} - \frac{i}{2\pi}\theta_a e^{w_a(t)}\left(t\left(1-t\right)\left[\theta_a,\theta_\xi\right]\right)\right)\,dt
\end{aligned}$$

yields (3.16). □

A fundamental property of the reduced normal operator is that its b-indicial symbol only depends on the base $\partial M/B$ and not on the cotangent variables, so $\operatorname{Ch}^{\mathrm{even}}(I_b(N))$ can be regarded as a form on ∂M. Thus the result of the lemma shows that the forms

$$\operatorname{Ch}\left(\mathbb{E}\right) - \operatorname{Ch}^{\mathrm{even}}\left(a\right) \in \mathcal{C}^\infty\left(\partial M;\Lambda^{\mathrm{even}}\right),\ \eta\left(N\right) \in \mathcal{C}^\infty\left(\mathbb{S}U;\Lambda^{\mathrm{odd}}\right)$$

define a class in $\mathcal{H}^{\mathrm{odd}}\left(\mathbb{S}U,\pi\right)$.

Proposition 3.5. *The (odd) relative Chern character for $K_c^1({}^0T^*_{\partial M}M/B)$ of the (zero) cotangent bundle of a compact manifold with boundary may be realized in terms of the reduced normal operators by*

(3.17) $$\operatorname{Ch}^{odd}(P) = \left(\operatorname{Ch}\left(\mathbb{E}\right) - \operatorname{Ch}^{even}[I(\mathrm{RN}(P))], \eta_{\mathrm{b}}(\mathrm{RN}(P))\right),$$

as a class in $\mathcal{H}^{odd}(S^*\partial M/B,\pi)$ for $P \in \operatorname{Id} + \Psi_0^{-\infty}(M/B;\mathbb{E})$ Fredholm on $L^2_0(M/B)$.

Remark. The proof of this proposition is more complicated than that of Proposition 3.2 since the isomorphism (3.12) does not allow us to represent a K-theory class by an operator that is trivial at infinity (see §2).

Proof. Since $\mathrm{Ch}^{\mathrm{odd}}(P)$ defines a class in $\mathcal{H}^{\mathrm{odd}}(S^*\partial M/B,\pi)$ it remains only to show that this is the relative Chern character. To this end, we will show that the diagram

$$\begin{tikzcd} \mathcal{K}^0_{-\infty,0}(M/B) \arrow[r, "\cong"'] \arrow[d, "\mathrm{Ch}"] & K^0_{\mathrm{c}}(T^*\partial M/B) \arrow[d, "\mathrm{Ch}"] \\ \mathcal{H}^{\mathrm{odd}}(S^*\partial M/B,\pi) \arrow[r, "\cong"'] & H^{\mathrm{even}}_c(T^*\partial M/B) \end{tikzcd} \tag{3.18}$$

commutes. It is convenient to represent the compactly supported K-theory of $T^*\partial M/B$ using a classifying group

$$K^0_{\mathrm{c}}(T^*\partial M/B) = \varinjlim \left[X; C^\infty\left((\mathbb{S}^1,1);(\mathrm{GL}(N),\mathrm{Id})\right)\right] = \left[X; G^{-\infty}\right],$$

where $G^{-\infty}$ is the group of invertible 'suspended' operators on a closed manifold that differ from the identity by a smoothing operator (see [20]).

The isomorphism at the top of (3.18) between the group of stable equivalence classes of invertible reduced normal families of perturbations of the identity and a standard presentation of $K^1_{\mathrm{c}}({}^0T^*_{\partial M}M/B)$ comes from the contractibility of the underlying semigroup of invertible operators on the interval. Namely, this allows the reduced normal family to be connected (after stabilization) to the identity through a curve of maps $A(t)$ from $S^*\partial M/B$ into the invertible b-operators of the form $\mathrm{Id}+A$, A of order $-\infty$ and non-trivial only at the one end of the interval. In fact we can easily arrange for the family $A(t)$ to be constant near the two endpoint of the parameter interval.

The indicial family of this curve initially only depends on the base variables $\partial M/B$ and the indicial parameter, so can be interpreted as fixing a homotopy class of smooth maps

$$\alpha : {}^0T^*\partial M/B \longrightarrow G^{-\infty}$$

and hence an element of $K^0_{\mathrm{c}}({}^0T^*\partial M/B)$.

The cohomology class of the resulting Chern character defines, via the map (1.26), an element of $\mathcal{H}^{\mathrm{odd}}(S^*\partial M/B,\pi)$. In this case, this map consists of passing from $\mathrm{Ch}(\alpha)$ back to $\mathrm{Ch}(A(t))$, decomposing this as

$$\mathrm{Ch}(A(t)) = \mathrm{Ch}(A_t) + \mathrm{Ch}'(A_t)\wedge dt$$

with both $\mathrm{Ch}(A_t)$ and $\mathrm{Ch}'(A_t)$ independent of dt, and then keeping

$$\left[\left(\mathrm{Ch}(A_0), -\int_0^\infty \mathrm{Ch}'(A_t)\ dt\right)\right] \in \mathcal{H}^{\mathrm{odd}}(S^*\partial X,\pi). \tag{3.19}$$

The normalization of the Chern character $\mathrm{Ch}^{\mathrm{even}}(a)$ is fixed by the requirement that it *be* the usual multiplicative map

$$\mathrm{Ch} : K^0(\partial M/B) \longrightarrow H^{\mathrm{even}}(\partial M/B).$$

Hence the first term in (3.19) coincides with $\mathrm{Ch}^{\mathrm{even}}(a)$ defined in (3.15), while the second term ω satisfies $d\omega = -\pi^*\mathrm{Ch}^{\mathrm{even}}(a)$. It follows that (3.19) and (3.17) define the same class in $\mathcal{H}^{\mathrm{odd}}(S^*\partial X,\pi)$. $\square$

General Fredholm zero operators. Again consider a fibration (1), where the fibers X are manifolds with boundary. The compactly supported K-theory of T^*M/B relative to the boundary can be represented by stable homotopy classes of Fredholm operators in the zero calculus. Thus each class is represented by a superbundle $\mathbb{E}$ and a pair of maps,

$$(\sigma, \mathcal{N}) \in \mathcal{C}^\infty \left(S^*M/B; \hom \mathbb{E}\right) \oplus \mathcal{C}^\infty \left(S^*\partial M/B; \Psi^0_{b,c}\left(\mathcal{I}; \mathbb{E}\right)\right)$$

as in (2.3). The vector bundle $W = T^*M/B$ has a trivial line sub-bundle at the boundary L (the normal bundle to the boundary) so $H^*_c\left(W^\circ\right)$ can be realized using forms on X, $\mathbb{S}W$, $\overline{L}$, and $\mathbb{S}U$ (with $U = W\big|_{\partial M}/L$). This will allow us to write a formula for the Chern character involving only $\mathbb{E}$, σ, $\mathcal{N}$ and a choice of graded connection on $\mathbb{E}$. To simplify the discussion the map σ restricted to the inward pointing end of L will be used to identify E^+ and E^- over the boundary. Then the connections can be chosen to be compatible with this identification and the result is that we can arrange

the Chern character of $\mathbb{E}$ restricted to the boundary is identically zero.

The main novelty in the construction of the explicit Chern character for general Fredholm families in the zero calculus involves the eta term coming from the reduced normal operator. This is now a doubly-regularized form, in the sense that the divergence at the boundary needs to be removed as before but there is also divergence coming from the fact that these operators are now not of trace class even locally in the interior of the interval. Indeed, an operator in the b,c calculus on the interval will be of trace class if and only if it has order less than -1 and its kernel vanishes at the boundary. This regularization is discussed in Appendix B (extending [18]) using complex powers of an admissible operator Q in the b,c calculus and a (total) boundary defining function x.

As discussed in Appendix A, the choice of a metric at the boundary trivializes the interval bundle over the boundary on which the reduced normal operator acts. In particular the reduced normal operator of an element of order 0 in the zero calculus then becomes a well-defined smooth map $\mathcal{N} : S^*\partial M/B \longrightarrow \Psi^0_{\mathrm{b,sc}}\left(\mathcal{I}; \mathbb{E}\big|_{\partial M}\right)$. The total symbol of Q is a function on the cotangent bundle of the interval $T^*\mathcal{I} = \mathcal{I}_r \times \mathbb{R}_\omega$ and it depends only on the cotangent variable ω, not on r.

Generalizing (3.14), for any element of the zero calculus with invertible reduced normal family (on L^2), set

$$\text{(3.20)} \quad \eta^{\mathrm{odd}}_{\mathrm{b,sc}}\left(\mathcal{N}\right) \\ = \frac{i}{2\pi} \int_0^1 \left(1-t\right)^R \mathrm{Tr}_{\mathrm{b,sc}} \left(\mathcal{N}^{-1}\left(\nabla\mathcal{N}\right) e^{\omega_{\mathcal{N}}(t)}\right) + t^R \, \mathrm{Tr}_{\mathrm{b,sc}} \left(\left(\nabla\mathcal{N}\right) e^{\omega_{\mathcal{N}}(t)} \mathcal{N}^{-1}\right) \, dt$$

where the inverse takes values in the large calculus.

The differential of (3.20) will involve the even Chern characters of the indicial families at the b and *cusp* ends. These are defined by

$$\mathrm{Ch}^{\mathrm{even}}\left(a\right) = \int_{\mathbb{R}} \mathrm{i}_{\partial_\xi} \widetilde{\mathrm{Ch}}_{Y\times\mathbb{R}}\left(a\right) \, d\xi$$

with $a = I_b\left(\mathcal{N}\right)$ or $a = I_c\left(\mathcal{N}\right)$. Initially one might expect an integrability issue because both of these are homogeneous of degree zero. However at least one of the

factors in the integrand will involve $\frac{\partial a}{\partial \xi}$, and since in either case a has an expansion

$$a \sim a_0 + a_1 \xi^{-1} + a_2 \xi^{-2} + \dots$$

as $\xi \to \infty$, the integrand must vanish to second order at infinity and hence is integrable.

Lemma 3.6. *For an element $A \in \Psi^0_0(X;\mathbb{E})$ with reduced normal operator $\mathcal{N}$: $X \longrightarrow \Psi^0_{\mathrm{b,sc}}\left(\mathcal{I}, \mathbb{E}\big|_{\partial X}\right)$ taking values in the L^2-invertible operators,*

$$\begin{aligned} d\eta^{odd}_{\mathrm{b,sc}}(\mathcal{N}) &= -\pi^* i^*_{\partial X} \operatorname{Ch}(\mathbb{E}) + \operatorname{Ch}^{even}(I_b(\mathcal{N})) - \operatorname{Ch}^{even}(I_c(\mathcal{N})) \\ &\operatorname{Ch}^{even}(I_b(\mathcal{N})) - \widehat{\nu}^L_* i^*_{\partial X} \widetilde{\operatorname{Ch}}(\sigma(A)) \end{aligned} \tag{3.21}$$

Proof. The computation in the proof of Lemma 3.4 shows that

$$\begin{aligned} d\eta^{\mathrm{odd}}_{\mathrm{b,sc}}(\mathcal{N}) = -\pi^* \operatorname{Ch}(\mathbb{E}) &+ \int_0^1 {}^R\operatorname{Tr}\left(\left[N e^{w_N(t)}, N^{-1}\right]\right)\, dt \\ &+ \frac{i}{2\pi}\int_0^1 {}^R\operatorname{Tr}\left(t(1-t)\left[N\left[\theta e^{w_N(t)}, \theta\right], N^{-1}\right]\right)\, dt, \end{aligned}$$

and we need only apply the trace-defect formula for ${}^R\operatorname{Tr}$ explained in Appendix B,

$$ {}^R\operatorname{Tr}([A,B]) = -\widehat{\operatorname{Tr}^{\sigma}}\left(\frac{BD_Q(A) + D_Q(A)B}{2}\right) + \widehat{\operatorname{Tr}^{\partial}}\left(\frac{BD_x(A) + D_x(A)B}{2}\right). \tag{3.22}$$

Notice that the principal symbol of $\mathcal{N}$ and the full symbol of Q^τ are independent of the interval variable $r \in \mathcal{I}$ and only depend on the cotangent variable, so the formula for the full symbol of the commutator $[A, Q(z)]$ shows that this operator is of order $-z-2$ and hence the first term in (3.22) vanishes. The second term can be written in terms of the indicial families at the b and c ends much like (3.13). Thus, at the b-end we get

$$\frac{i}{2\pi}\int_0^1 \left[{}^R\!\!\int_{\mathbb{R}} \operatorname{tr}\left(\theta_\xi e^{w_a(t)}\right)\, d\xi + \frac{i}{2\pi} t(1-t)\, {}^R\!\!\int_{\mathbb{R}} \operatorname{tr}\left(-\theta_\xi\left[\theta_a e^{w_a(t)}, \theta_a\right]\right)\, d\xi \right] dt,$$

which, as in the proof of Lemma 3.4, equals $\int_{\mathbb{R}} \mathrm{i}_{\partial_\xi} \widetilde{\operatorname{Ch}}(I_b(\mathcal{N}))$ and similarly at the cusp end. This proves the first line in (3.21). The second line follows since, on the one hand, we required that $i^*_{\partial X}\operatorname{Ch}(\mathbb{E})$ vanish, and, on the other, the cusp indicial family is given by

$$I_c(\mathcal{N}(y,\eta))(\xi) = {}^0\sigma(A)(0, y, \xi, \eta).$$

□

With this lemma we have all of the ingredients for representing the Chern character of a relative K-theory class in terms of a zero pseudodifferential operator using the description of the relative cohomology from §1.6. Recall that the chain complex is

$$\mathcal{Z}_k = \mathcal{C}^\infty\left(X;\Lambda^k\right) \oplus \left(\mathcal{C}^\infty\left(\mathbb{S}W;\Lambda^{k-1}\right) \oplus \mathcal{C}^\infty\left(\overline{L};\Lambda^{k-1}\right)\right) \oplus \mathcal{C}^\infty\left(\mathbb{S}U;\Lambda^{k-3}\right)$$

$$D = \begin{pmatrix} d & & & \\ -\pi^*_{\mathbb{S}W} & -d & & \\ -\pi^*_{\overline{L}} i^*_\partial & & -d & \\ & -\nu^{\mathbb{S}W}_* & \pi^*_{\mathbb{S}U}\nu^L_* & d \end{pmatrix}$$

(with a compatibility condition at $L_\pm$).

Theorem 3.7.
a) The element of $\mathcal{C}_{even}$,

$$\mathrm{Ch}\,(W,\mathbb{E},\sigma,\mathcal{N}) = \Big(\mathrm{Ch}\,\mathbb{E}, \Big(\widetilde{\mathrm{Ch}}\,a, \widetilde{\mathrm{Ch}}\,I_b\,(\mathcal{N})\Big), -\eta\,(\mathcal{N})\Big)\,, \tag{3.23}$$

is D-closed and represents the Chern character of the K-theory class defined by $(W,\mathbb{E},\sigma,\mathcal{N})$.

b) Suppose that $A \in \Psi^0_0\,(M/B;\mathbb{E})$ is a fully elliptic family of 0-pseudodifferential operators acting on the fibers of a fibration $M \to B$, then the Chern character of the index bundle of A is given by

$$\begin{aligned}\mathrm{Ch}\,(\mathrm{Ind} A)&\\ &= \int_{S^*M/B,S^*\partial M/B} \mathrm{Ch}\,(T^*M/B,\mathbb{E},\sigma\,(A)\,,\mathcal{N}\,(A)) \wedge \mathrm{Td}\,(M/B)\\ &= \int_{{}^0S^*M/B} \widetilde{\mathrm{Ch}}\,(\sigma\,(A)) \wedge \mathrm{Td}\,(M/B) - \int_{S^*\partial M/B} \eta\,(\mathcal{N}\,(A)) \wedge \mathrm{Td}\,(\partial M/B)\,.\end{aligned} \tag{3.24}$$

Proof. Showing that (3.23) is D-closed is equivalent to showing

$$d\,\mathrm{Ch}\,\mathbb{E} = 0, \quad d\,\widetilde{\mathrm{Ch}}\,a = -\pi^*\,\mathrm{Ch}\,\mathbb{E},$$
$$d\,\widetilde{\mathrm{Ch}}^{ev}\,I_b\,(\mathcal{N}) = 0, \quad d\eta\,(\mathcal{N}) = -\nu_*^{\mathbb{S}W}\,\widetilde{\mathrm{Ch}}(\sigma(A)) + \pi_{\mathbb{S}U}^*\nu_*^{\overline{L}}\,\widetilde{\mathrm{Ch}}(I_b(\mathcal{N}))$$

the last one of which is Lemma 3.6 (since $\nu_*^{\overline{L}}\,\widetilde{\mathrm{Ch}}(I_b(\mathcal{N})) = \mathrm{Ch}^{\mathrm{even}}(I_b(\mathcal{N})))$ while the other three follow as in the previous section.

Also as in the previous sections, the forms themselves are invariant under stabilization and bundle isomorphism while Lemma 1.2 shows that the cohomology class they define is homotopy invariant. It follows that we can change the 0-operator whose normal operators define a and $\mathcal{N}$ as long as we stay in the same K-theory class. As explained in §2 there is a representative of this class with $\mathcal{N}$ equal to the identity and a equal to the identity near the boundary. For this representative the map above clearly coincides with the Chern character and hence this is true for any representative.

In the same way (3.24) follows from homotopy invariance and reduction to the case that the operator is trivial near the boundary. □

3.3. **Transmission families index theorem.** In the same situation as above any class in $K_{\mathrm{c}}(T^*M^\circ/B)$ can also be represented by a fully elliptic operator or order and type zero in Boutet de Monvel's transmission calculus. With $W = T^*M/B$, L equal to the normal bundle to the boundary, and $U = W\big|_{\partial M}/L$ a K-theory class is represented by a superbundle over M, $\mathbb{E}$, a superbundle over ∂M, $\mathbb{F}$, and two maps: a family of isomorphisms $a \in \mathcal{C}^\infty\,(\mathbb{S}W;\hom\,(\mathbb{E}))$ and a family of Wiener-Hopf operators $N \in \mathcal{C}^\infty\,(\mathbb{S}U;\Psi^*_{WH}\,(L;(H^+\otimes\mathbb{E})\oplus\mathbb{F}))$. We use this data, and a choice of graded connections on $\mathbb{E}$ and $\mathbb{F}$, to represent the Chern character using the description of the relative cohomology from §1.6 in terms of forms on X, $\mathbb{S}W$, ∂X, and $\mathbb{S}U$. As in the previous section we will assume that

> the graded connection on $\mathbb{E}$ is chosen so that the restriction of $\mathrm{Ch}\,(\mathbb{E})$ to the boundary is identically zero.

This description of the Chern character and the resulting index formula are treated by Fedosov in [6, §III.4]; as in Proposition 3.1 above, we reinterpret this formula in an appropriate formulation of relative cohomology.

The family of Wiener-Hopf operators plays much the same role as the family of b, c operators in the previous section. Were these trace-class it would be natural to use their Chern character in our constructions. However, they are not trace-class so we are forced to use a renormalized trace and we refer to the resulting form as an 'eta' form. In this case the renormalized trace was defined by Fedosov as follows. With $N = \begin{pmatrix} h^+p+b & k \\ t & q \end{pmatrix}$ as in (2.5), N is trace-class if and only if $h^+p = 0$ so we define tr' by ignoring this term [6, §4]

$$\operatorname{tr}'(N) = \operatorname{tr} q + \frac{1}{2\pi} \int_{\mathbb{R}}^{+} \operatorname{tr} b(\xi, \xi)\, d\xi, \tag{3.25}$$

where in the first term we use $\operatorname{tr} : \hom(\mathbb{F}) \to \mathbb{R}$ applied to q, and in the second $\operatorname{tr} : \hom(\mathbb{E}) \to \mathbb{R}$ applied to the integral kernel of b at the point (ξ, ξ).

This renormalized trace is not an actual trace, but instead satisfies a trace-defect formula [6, Lemma 2.1]

$$\operatorname{tr}'[N_1, N_2] = -\frac{i}{2\pi} \int_{\mathbb{R}}^{+} \operatorname{tr}\left(\frac{\partial p_1(\xi)}{\partial \xi} p_2(\xi) \right) d\xi. \tag{3.26}$$

It is clear from this formula that if either N_1 or N_2 is 'singular' (i.e., $p_1 = 0$ or $p_2 = 0$) then $\operatorname{tr}'[N_1, N_2] = 0$.

Denote the chosen graded connections on $\mathbb{E}$ and $\mathbb{F}$ by ∇ and ∇^{∂} respectively. Using both of these we define a connection on each of the bundles $(H^+ \otimes E_{\pm}) \oplus F_{\pm}$ acting trivially on H^+, we denote the resulting graded connection again by ∇. Notice that $d \operatorname{tr}'(N) = \operatorname{tr}'(\nabla N)$.

The trace-defect formula (3.26) is formally identical to that of the b-trace (3.13) on smoothing operators. So if we define $\eta(N)$ as an element of $\mathcal{C}^{\infty}\left(\mathbb{S}U; \Lambda^{\mathrm{odd}}\right)$ by

$$\begin{gathered}\eta(N) = -\frac{1}{2\pi i} \int_0^1 (1-t) \operatorname{tr}'\left(N^{-1}(\nabla N) e^{w_N(t)} \right) + t \operatorname{tr}'\left((\nabla N) e^{w_N(t)} N^{-1} \right) dt \\ \text{where } w_N(t) = (1-t)\omega_+ + tN^{-1}\omega_- N + \frac{1}{2\pi i} t(1-t)(N^{-1}\nabla N)^2\end{gathered} \tag{3.27}$$

then the computations in the proof of Lemma 3.4 apply verbatim to compute $d\eta(N)$ and we conclude that

$$d\eta(N) = -\pi_{\partial}^* \operatorname{Ch}(\mathbb{E}_{\partial} \oplus \mathbb{F}) + \nu_*^L i_{\partial}^* \widetilde{\operatorname{Ch}}(a), \tag{3.28}$$

where, with $-2\pi i \omega'_{\pm}$ equal to the curvature of ∇ on $(H^+ \otimes \mathbb{E}) \oplus \mathbb{F}$,

$$\operatorname{Ch}(\mathbb{E}_{\partial} \oplus \mathbb{F}) = \operatorname{tr} e^{\omega'_+} - \operatorname{tr}^{\omega'_-} \in \mathcal{C}^{\infty}(\partial X; \Lambda^{\mathrm{even}})$$

and as usual

$$\begin{gathered}\widetilde{\operatorname{Ch}}(a) = -\frac{1}{2\pi i} \int_0^1 \operatorname{tr}\left(a^{-1}(\nabla a) e^{w_a(t)} \right) dt \in \mathcal{C}^{\infty}\left(\mathbb{S}W; \Lambda^{\mathrm{odd}}\right), \\ \text{with } w_a(t) = (1-t)\omega_+ + ta^{-1}\omega_- a + \frac{1}{2\pi i} t(1-t)(a^{-1}\nabla a)^2.\end{gathered} \tag{3.29}$$

It follows that with the chain space from §1.6

$$\mathcal{T}^k = \mathcal{C}^\infty(X;\Lambda^k) \oplus \left(\mathcal{C}^\infty_\pm(\mathbb{S}W;\Lambda^{k-1}) \oplus \mathcal{C}^\infty(\partial X;\Lambda^{k-2})\right) \oplus \mathcal{C}^\infty(\mathbb{S}U;\Lambda^{k-3})$$

$$D_{\mathcal{T}} = \begin{pmatrix} d & & \\ \phi_1 & -d & \\ & \phi_2 & d \end{pmatrix}, \phi_1 = \begin{pmatrix} -\pi^*_{\mathbb{S}W} \\ i^*_{\partial X} \end{pmatrix}, \phi_2 = \left(\nu^{\mathbb{S}W}_*, \pi^*_{\mathbb{S}U}\right),$$

the forms

$$\operatorname{Ch}(\mathbb{E},\mathbb{F},a,N) = \left(\operatorname{Ch}(\mathbb{E}), \left(\widetilde{\operatorname{Ch}}(a), -\operatorname{Ch}(\mathbb{E}_\partial \oplus \mathbb{F})\right), -\eta(N)\right) \tag{3.30}$$

define a relative cohomology class.

Theorem 3.8.

a) $\operatorname{Ch}(\mathbb{E},\mathbb{F},a,N)$ *is a D-closed element of $\mathcal{T}^{even}$ and represents the Chern character of the K-theory class defined by $(W,\mathbb{E},\mathbb{F},a,N)$.*

b) Suppose that $\mathcal{A} \in \Psi^0_{\mathrm{tm}}(M/B;\mathbb{E};\mathbb{F})$ is a fully elliptic family of transmission pseudodifferential operators acting on the fibers of a fibration $M \to B$, then the Chern character of the index bundle of $\mathcal{A}$ is given by

$$\begin{aligned} &\operatorname{Ch}(\operatorname{Ind}(\mathcal{A})) \\ &\quad= \int_{S^*M/B, S^*\partial M/B} \operatorname{Ch}(T^*M/B,\mathbb{E},\mathbb{F},\sigma(\mathcal{A}),N(\mathcal{A})) \wedge \operatorname{Td}(X,\partial X) \\ &\quad= \int_{S^*M/B} \widetilde{\operatorname{Ch}}(\sigma(\mathcal{A})) \wedge \operatorname{Td}(M/B) - \int_{S^*\partial M/B} \eta(N(\mathcal{A})) \wedge \operatorname{Td}(\partial M/B). \end{aligned} \tag{3.31}$$

Proof. As in the previous section, we have shown that (3.30) defines a D-closed form, Lemma 1.2 shows that it depends only on the K-theory class and evaluating it on a representative that is trivial at the boundary shows that (3.30) is the usual Chern character and reduces (3.31) to the Atiyah-Singer families index theorem. □

Appendix A. Normal operator

We review the reduced normal operator in the zero calculus.

Zero differential operators are elements of the enveloping algebra of the vector fields that vanish at the boundary, $\mathcal{V}_0$. Thus if x is a boundary defining function and y_i are local coordinates along the boundary of X, we have

$$\mathcal{V}_0 = \operatorname{Span}_{\mathcal{C}^\infty(X)}\langle x\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_n}\rangle,$$

$$\text{and } P \in \operatorname{Diff}^k_0(M;E,F) \iff P = \sum_{j+|\alpha|\le k} a_{j,\alpha}(x,y)(x\partial_x)^j(x\partial_y)^\alpha,$$

where the coefficients, $a_{j,\alpha}$, are section of the homomorphism bundle $\hom(E,F)$.

It is convenient to study these operators by viewing their Schwartz kernels as defined on a compactification of $(M^\circ)^2$ different from $\overline{M}^2$. We construct M^2_0 from $\overline{M}^2$ by replacing the diagonal at the boundary, $\operatorname{diag}_{\partial M}$, with its inward pointing spherical normal bundle,

$$M^2_0 = (\overline{M}^2 \setminus \operatorname{diag}_{\partial M}) \bigsqcup (S^+\operatorname{diag}_{\partial M})$$

and endowing the result with the smallest smooth structure including smooth functions on $\overline{M}^2$ and polar coordinates around $\operatorname{diag}_{\partial M}$ – a process known as 'blowing-up $\operatorname{diag}_{\partial M}$ in $\overline{M}^2$'. The diagonal lifts from the interior of M^2 to a submanifold, diag_0,

of M_0^2 (a p-submanifold in the sense of [14]). The set of 'Dirac sections' along the diagonal on M^2 coincides with the space of Schwartz kernels of arbitrary differential operators on M, the Schwartz kernels of zero differential operators are precisely those that lift to Dirac sections of M_0^2 along diag_0.

The passage from zero differential operators to zero pseudodifferential operators is a micro-localization in that we enlarge the space of Schwartz kernels from Dirac sections along diag_0 to distributions on M_0^2 with conormal singularities along diag_0, i.e., $I^s(M_0^2, \mathrm{diag}_0)$ in the notation of [14]. We further demand that at each boundary face of M_0^2 the Schwartz kernel have a classical asymptotic expansion in the corresponding boundary defining function and its logarithm. We refer to [9], [10] for the details.

We can use the expansions at each of the boundary faces to define 'normal operators' by restricting to the leading term at that face. The zero diagonal meets the boundary at the front face (the face introduced by the blow-up of $\mathrm{diag}_{\partial M}$) and the normal operator at this face is known as *the* normal operator. Directly from the definition it is clear that the restriction of the kernel of a zero pseudodifferential operator of order s to the front face is a distribution in

$$I^{s+\frac{1}{4}}(S^+\mathrm{diag}_{\partial M}, \mathrm{diag}_0 \cap S^+\mathrm{diag}_{\partial M}) \cong I^{s+\frac{1}{4}}(\partial M \times \mathbb{R}^{m-1} \times \mathbb{R}^+, \partial M \times \{(0,1)\}).$$

It can be shown that the kernels of normal operators define, for each $q \in \partial M$, a distribution on $T_q M^+ \cong \mathbb{R}^{m-1} \times \mathbb{R}^+$ that is invariant with respect to the affine group action

$$\begin{aligned} (\mathbb{R}^{m-1} \times \mathbb{R}^+)^2 &\longrightarrow \mathbb{R}^{m-1} \times \mathbb{R}^+ \\ ((a,b),(c,d)) &\longmapsto (a + bc, bd) \end{aligned}$$

and that the normal operator is a homomorphism in that the kernel of the normal operator of the composition of two operators is the convolution of the normal operators. Thus analyzing the normal operator is tantamount to harmonic analysis of the affine group on $\mathbb{R}^{m-1} \times \mathbb{R}^+$.

The invariance with respect to the affine group action suggests studying the normal operator by first taking the Fourier transform in $\mathbb{R}^{m-1}$ and then exploiting dilation invariance with respect to the $\mathbb{R}^+$ variable. The result is a family of operators acting on the normal bundle to the boundary and parametrized by the cosphere bundle to the boundary. The choice of a trivialization of the normal bundle allows us to identify this with an element of

$$\mathcal{C}^\infty(\mathbb{S}^*\partial M, \Psi_b^s(\mathbb{R}^+)).$$

Finally let x be the identity map $\mathbb{R}^+ \to \mathbb{R}^+$, we compactify $\mathbb{R}^+$ by introducing $\frac{1}{x}$ as a boundary defining function for the 'point at ∞'. After performing these transformations on the normal operator, and identifying the compactification of $[0,\infty)$ with $[0,1]$, the normal operator is an operator of b-type at the 0 end and an operator of sc-type at the 1 end, with a singularity of order $-s$. This family is known as the *reduced normal operator*, e.g. ,

$$A \in \Psi_0^0(X) \implies \mathcal{N}(A) \in \mathcal{C}^\infty(\mathbb{S}^*\partial M, \Psi^0_{\mathrm{b,sc}}([0,1])).$$

The distributional kernels of elements of the $\mathrm{b,sc}$ can be conveniently described on the $\mathrm{b,sc}$ double space, $[0,1]^2_{\mathrm{b,sc}}$ obtained from $[0,1]^2$ in two steps. In the first, we blow-up the corners $\{(0,0),(1,1)\}$ obtaining two new boundary faces, one bf_0 replacing $\{(0,0)\}$ and one bf_1 replacing $\{(1,1)\}$. In the second step, we blow-up

the intersection of bf_1 with the closure of the diagonal of $(0,1)^2$; we refer to the resulting boundary face as cf. The closure of the diagonal of $(0,1)^2$ in $[0,1]^2_{\mathrm{b,sc}}$ is referred to as $\mathrm{diag}_{\mathrm{b,sc}}$. Pseudo-differential operators of $\mathrm{b,sc}$ type on $[0,1]$ are those whose Schwartz kernels are distributions on $[0,1]^2_{\mathrm{b,sc}}$ conormal with respect to $\mathrm{diag}_{\mathrm{b,sc}}$ and otherwise smooth that vanish to infinite order at every boundary face that does not meet $\mathrm{diag}_{\mathrm{b,sc}}$.

These operators are sometimes known as the *small* calculus of $\mathrm{b,sc}$ operators and there is also an associated *large* calculus of $\mathrm{b,sc}$ pseudodifferential operators where the Schwartz kernels need not vanish at the boundary faces that share a corner with bf_0 (known as the 'side faces'). It is standard (see [15], [10]) that parametrices and generalized inverses of $\mathrm{b,sc}$ pseudodifferential operators, when these exist, are elements of the large calculus of $\mathrm{b,sc}$ operators.

A complete metric on the interior of $[0,1]_x$ is of $\mathrm{b,sc}$ type if it takes the form $\frac{dx^2}{x^2}$ near $x=0$ and the form $\frac{dx^2}{(1-x)^4}$ near $x=1$. The corresponding space of square-integrable functions is denoted $L^2_{\mathrm{b,sc}}([0,1])$. Elements of the small calculus of $\mathrm{b,sc}$ operators of order zero define bounded operators on $L^2_{\mathrm{b,sc}}([0,1])$ as do elements of the large calculus, so long as they vanish to order greater than -1 at each of the side faces. Thus it makes sense to compose these operators and the references cited above show that the composition is given again by an element of, respectively, the small or large $\mathrm{b,sc}$ calculus.

Appendix B. Residue traces

We review the residue traces on algebras of pseudodifferential operators. These are used to define Chern and eta forms. We start by presenting the case of closed manifolds, before moving on to manifolds with boundary and finally operator valued forms. We refer the reader to [18] for the proofs of these statements for the algebra of cusp operators. The proofs consist of formal algebraic manipulations and hence hold verbatim for many other calculi – we shall need these results for the b,sc algebra in section 3.2

On a closed manifold, X, the Guillemin-Wodzicki residue trace, $\widehat{\mathrm{Tr}^{\sigma}}$, is the unique trace on $\Psi^*(X)$, and has two well-known expressions

$$\widehat{\mathrm{Tr}^{\sigma}}(A) = \int_{S^*X} \sigma_{(-n)}(A) = \operatorname*{Res}_{\tau=0} \mathrm{Tr}\left(Q^{-\tau}A\right). \tag{B.1}$$

The first expression (due to Wodzicki) involves the term in the expansion of the full symbol of $A \in \Psi^{\mathbb{Z}}(X)$ that is homogeneous of order $-n$, $\sigma_{(-n)}$. Although $\sigma_{(-n)}(A)$ is not invariantly defined, its integral is (notice that scale invariance as a density forces the homogeneity). The second expression (due to Guillemin) involves the choice of an *admissible* first order operator Q, which we shall understand to mean that Q is invertible, essentially self-adjoint, and positive. The function $\tau \mapsto \mathrm{Tr}(Q^{-\tau}A)$ is holomorphic on a half-plane and can be meromorphically extended to the whole complex plane. The pole at $\tau=0$ is at worst a simple pole and the residue is independent of the choice of Q.

The first expression above shows that the Guillemin-Wodzicki residue vanishes on operators of order less than $-n$, the space of trace-class operators. For each choice of admissible operator Q, we can extend the trace from trace-class operators

to a *renormalized* trace on all of Ψ^* by

$$\text{(B.2)} \qquad {}^R\operatorname{Tr}(A) = \underset{\tau=0}{\operatorname{FP}} \operatorname{Tr}\left(Q^{-\tau}A\right) = \lim_{\tau\to 0}\left(\operatorname{Tr}\left(Q^{-\tau}A\right) - \frac{1}{\tau}\widehat{\operatorname{Tr}}^{\sigma}(A)\right).$$

This renormalized trace does depend on the choice of Q, indeed

$$ {}^R\operatorname{Tr}_{Q_1}(A) - {}^R\operatorname{Tr}_{Q_2}(A) = -\widehat{\operatorname{Tr}^{\sigma}}\left(A\log\left(Q_1/Q_2\right)\right)$$

where we define

$$\text{(B.3)} \qquad \log\left(Q_1/Q_2\right) = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0}\left(Q_1^{\tau}Q_2^{-\tau}\right).$$

We also point out that ${}^R\operatorname{Tr}$ is not really a trace, since it does not vanish on commutators:

$$ {}^R\operatorname{Tr}\left([A,B]\right) = -\widehat{\operatorname{Tr}^{\sigma}}\left(D_Q(A)\,B\right),$$

with D_Q the derivation on $\Psi^*(X)$ defined by

$$D_Q(A) = \left.\frac{\partial}{\partial\tau}\right|_{\tau=0} Q^{\tau}AQ^{-\tau}.$$

In this subsection we review the extension by Melrose-Nistor [18] (see also [8], [22]) of the Guillemin-Wodzicki residue trace to operator algebras on manifolds with boundary.

We define, for each choice of bdf x and admissible first order positive $Q \in \Psi^1_{\mathrm{b,sc}}(M;E)$, 'trace' functionals on the algebra $\Psi^*_{\mathrm{b,sc}}(M;E)$. By a simple extension of the results of Piazza [23], complex powers of Q can be defined as elements of the *large* calculus of $\mathrm{b,sc}$ operators (described in §A). Assume for the moment that A and B are elements of the small calculus of $\mathrm{b,sc}$ operators. It is shown in [18, Lemma 4] that the function

$$\mathcal{Z}(A;z,\tau) = \frac{1}{2}\left[\operatorname{Tr}\left(x^zQ^{-\tau}A\right) + \operatorname{Tr}\left(Q^{-\tau}x^zA\right)\right]$$

is holomorphic for $\operatorname{Re} z, \operatorname{Re}\tau >> 0$ and extends meromorphically to $\mathbb{C}^2$ with at most simple poles in z or τ. We use the expansion at zero to define the following functionals

$$\text{(B.4)} \quad \mathcal{Z}(A;z,\tau) \sim \frac{1}{z\tau}\operatorname{Tr}^{\partial,\sigma}(A) + \frac{1}{z}\widehat{\operatorname{Tr}^{\partial}}(A) + \frac{1}{\tau}\widehat{\operatorname{Tr}^{\sigma}}(A) + {}^R\operatorname{Tr}(A) + \mathcal{O}(z,\tau).$$

Among these functionals, only $\operatorname{Tr}^{\partial,\sigma}$ is a trace (vanishes on all commutators) and is independent of the choice of x and Q. We note that the residue trace functionals could be defined using $\operatorname{Tr}\left(x^zQ^{-\tau}A\right)$ or $\operatorname{Tr}\left(Q^{-\tau}x^zA\right)$. This follows from the fact that

$$\operatorname{Tr}\left(x^zQ^{-\tau}A\right) - \operatorname{Tr}\left(Q^{-\tau}x^zA\right)$$

is regular at $z = 0$, $\tau = 0$. The renormalized trace, though, would *a priori* be different.

On the ideal $x^{\infty}\Psi^*_{\mathrm{b,sc}}(X)$, the functionals $\operatorname{Tr}^{\partial,\sigma}$ and $\widehat{\operatorname{Tr}^{\partial}}$ both vanish, the functional $\widehat{\operatorname{Tr}^{\sigma}}$ is a trace and is given by the expressions (B.1). Similarly, on the ideal $\Psi^{-\infty}_{\mathrm{b,sc}}(X)$, the functionals $\operatorname{Tr}^{\partial,\sigma}$ and $\widehat{\operatorname{Tr}^{\sigma}}$ both vanish, the functional $\widehat{\operatorname{Tr}^{\partial}}$ is a trace

and has an expression similar to (B.1). Indeed, recall that, by Lidskii's theorem, if A is a trace-class operator with Schwartz kernel $\mathcal{K}_A$, then

$$\mathrm{Tr}\,(A) = \int_X \mathcal{K}_A\big|_{\mathrm{diag}}$$

(note that $\mathcal{K}\big|_{\mathrm{diag}}$ is a density). For a general operator $A \in \Psi^{-\infty}_{\mathrm{b,sc}}(X)$, having chosen a bdf x, we can expand $\mathcal{K}\big|_{\mathrm{diag}}$ in its 'Taylor's series' at $x = 0$

$$\mathcal{K}_A \sim \sum_{\ell \geq 0} x^\ell \mathcal{K}_\ell, \tag{B.5}$$

and one of these terms will be invariant under the rescaling $x \mapsto \lambda x$, say $x^r \mathcal{K}_r$ (since $\mathcal{K}_A$ is a singular density at $x = 0$, this is not $\mathcal{K}_0$). It is easy to see that the residue at $z = 0$ of

$$\mathrm{Tr}\,(x^z A) = \int_X x^z \mathcal{K}_A\big|_{\mathrm{diag}}$$

is given by the partial integral of $\mathcal{K}_r$,

$$\widehat{\mathrm{Tr}^\partial}\,(A) = \int_{\partial X} \mathcal{K}_r\big|_{\partial X}.$$

Just as for the Guillemin-Wodzicki residue, the density $\mathcal{K}_r\big|_{\partial X}$ depends on the choice of x, but its integral along the boundary does not.

For operators not in these ideals, $\widehat{\mathrm{Tr}^\sigma}$ is obtained by taking a 'residue in the symbol' and 'renormalization at the boundary' meaning, for instance, that we renormalizw the integral occuring in the first expression in (B.1). Where by renormalized integral of a density μ, we mean

$${}^R\!\!\int \mu = \underset{z=0}{\mathrm{FP}} \int x^z \mu.$$

Similarly, $\widehat{\mathrm{Tr}^\partial}$ is obtained by taking a 'residue at the boundary' and 'renormalization in the symbol'. So, for instance, expand the operator's kernel as in (B.5) as a distribution to pick out $\mathcal{K}_r$ (residue at the boundary), this defines an pseudo-differential operator over the boundary and then its renormalized trace over the boundary defined as in (B.2) equals $\widehat{\mathrm{Tr}^\partial}$. The common residue, i.e., the 'residue at the boundary' and 'residue in the symbol', is given by the functional $\mathrm{Tr}^{\partial,\sigma}$.

The functional $\widehat{\mathrm{Tr}^\sigma}$ is independent of Q just as in the case of a closed manifold, but does depend on the choice of x via [18, Lemma 9]

$$\widehat{\mathrm{Tr}^\sigma_{x_1}}\,(A) - \widehat{\mathrm{Tr}^\sigma_{x_2}}\,(A) = \mathrm{Tr}^{\partial,\sigma}\left(\log\left(x'/x\right) A\right).$$

Similarly, $\widehat{\mathrm{Tr}^\partial}$ is independent of the choice of x, but depends on the choice of Q via [18, Lemma 11]

$$\widehat{\mathrm{Tr}^\partial_{Q_1}}\,(A) - \widehat{\mathrm{Tr}^\partial_{Q_2}}\,(A) = -\,\mathrm{Tr}^{\partial,\sigma}\left(\log\left(Q_1/Q_2\right) A\right)$$

with $\log\left(Q_1/Q_2\right)$ defined by (B.3). It follows that the functional $\mathrm{Tr}^{\partial,\sigma}$ is independent of both the choice of x and the choice of Q.

To analyze the behavior of these functionals on commutators, notice that

$$\begin{aligned} x^z Q^{-\tau}[A,B] &= x^z Q^{-\tau}\left(A - Q^{\tau} A Q^{-\tau}\right) B \\ &\qquad - x^z Q^{-\tau} B\left(A - x^z A x^{-z}\right) + \left[x^z A x^{-z}, x^z Q^{-\tau} B\right] \\ &= -x^z Q^{-\tau}\left(\tau D_Q(A) + \mathcal{O}\left(\tau^2\right)\right) B \\ &\qquad + x^z Q^{-\tau} B\left(z D_x(A) + \mathcal{O}\left(z^2\right)\right) + \left[x^z A x^{-z}, x^z Q^{-\tau} B\right] \end{aligned}$$

where

$$\begin{aligned} D_x(A) &= \left.\frac{\partial}{\partial z}\right|_{z=0}\left(x^z A x^{-z}\right) =: [\log x, A], \\ D_Q(A) &= \left.\frac{\partial}{\partial \tau}\right|_{\tau=0}\left(Q^{\tau} A Q^{-\tau}\right) =: [\log Q, A]. \end{aligned}$$

and since $\operatorname{Tr}\left(x^z Q^{-\tau} A\right)$ has only simple poles in z and τ the terms with $\mathcal{O}\left(\tau^2\right)$ and $\mathcal{O}\left(z^2\right)$ do not affect the finite part at zero, i.e. $\operatorname{Tr}\left(x^z Q^{-\tau}[A,B]\right)$ has the same finite part at zero as $\operatorname{Tr}\left(x^z Q^{-\tau}\left(-\tau D_Q(A) B + z B D_x(A)\right)\right)$.

A similar computation shows that $\operatorname{Tr}\left(Q^{-\tau} x^z [A,B]\right)$ has the same finite part at zero as $\operatorname{Tr}\left(Q^{-\tau} x^z\left(z D_x(A) B - \tau B D_Q(A)\right)\right)$. This shows that the expansion of $2\mathcal{Z}([A,B];z,\tau)$ at $z=0, \tau=0$ is

$$\begin{aligned} \text{(B.6)}\quad & -\frac{2}{z}\operatorname{Tr}^{\partial,\sigma}\left(B D_Q(A)\right) + \frac{2}{\tau}\operatorname{Tr}^{\partial,\sigma}\left(B D_x(A)\right) \\ &\quad - \widehat{\operatorname{Tr}^{\sigma}}\left(B D_Q(A) + D_Q(A) B\right) + \widehat{\operatorname{Tr}^{\partial}}\left(B D_x(A) + D_x(A) B\right) \\ & -\frac{\tau}{z}\widehat{\operatorname{Tr}^{\partial}}\left(B D_Q(A) + D_Q(A) B\right) + \frac{z}{\tau}\widehat{\operatorname{Tr}^{\sigma}}\left(B D_x(A) + D_x(A) B\right) + \mathcal{O}(\tau, z). \end{aligned}$$

Notice that if $[A,B]$ is in $\Psi^{-\infty}_{\mathrm{b,sc}}(M;E)$ then those terms in (B.6) with $\frac{1}{z}$ must vanish, while if it is in $x^{\infty}\Psi^{*}_{\mathrm{b,sc}}(M;E)$ then those terms with $\frac{1}{\tau}$ must vanish. This observation can be used, together with Calderon's formula for the index, to obtain a residue trace formula for the index of a Fredholm operator, see [18], [22].

For any $\lambda > 1$ notice that

$$\widehat{\mathcal{Z}}(A;z) = \frac{\mathcal{Z}(A;z,\lambda z) + \mathcal{Z}\left(A;z,\frac{1}{\lambda}z\right) - \left(\lambda + \frac{1}{\lambda}\right)\mathcal{Z}(A;z,z)}{2 - \left(\lambda + \frac{1}{\lambda}\right)}$$

coincides with the usual trace if A is trace-class, and from (B.6) satisfies

$$\begin{aligned} \text{(B.7)}\quad & \widehat{\mathcal{Z}}([A,B];z) \sim -\frac{1}{z}\operatorname{Tr}^{\partial,\sigma}\left(B D_Q(A)\right) \\ &\quad - \widehat{\operatorname{Tr}^{\sigma}}\left(\frac{B D_Q(A) + D_Q(A) B}{2}\right) + \widehat{\operatorname{Tr}^{\partial}}\left(\frac{B D_x(A) + D_x(A) B}{2}\right) + \mathcal{O}(z), \end{aligned}$$

near $z=0$. Thus the renormalized trace of a commutator is given by the second line in (B.7) at $z=0$,

(B.8)

$${}^{R}\operatorname{Tr}([A,B]) = -\widehat{\operatorname{Tr}^{\sigma}}\left(\frac{B D_Q(A) + D_Q(A) B}{2}\right) + \widehat{\operatorname{Tr}^{\partial}}\left(\frac{B D_x(A) + D_x(A) B}{2}\right).$$

Finally we point out that the same formula holds for elements of the *large* calculus of $\mathrm{b,sc}$ operators. Indeed, we can choose a sequence of smooth, non-negative functions $\phi_k \in \mathcal{C}^{\infty}([0,1]^2_{\mathrm{b,sc}})$ that are identically equal to one in a neighborhood of the lifted diagonal $\mathrm{diag}_{\mathrm{b,sc}}$, vanish to infinite order at any boundary face that does

not meet $\mathrm{diag}_{\mathrm{b,sc}}$, converge uniformly to the constant function 1, and, in a collar neighborhood of the boundary face bf_0 are independent of the boundary defining function. Denote the right-hand-side of (B.8) by $\mathcal{F}(A,B)$ and define the operators A_k and B_k by multiplying the distributional kernels of A and B respectively by ϕ_k. Then A_k and B_k are in the small calculus and we point out that

$$^R\operatorname{Tr}([A,B]) = \lim_k {}^R\operatorname{Tr}([A_k,B_k]) = \lim_k \mathcal{F}(A_k,B_k) = \mathcal{F}(A,B),$$

since, for instance, the symbol of A_k coincides with that of A and the expansion of A_k at bf_0 is the same as that of A with the coefficients multiplied by ϕ_k.

Appendix C. Operator valued forms

We will make use of forms on the vector space $\Psi^*_{\mathrm{b,sc}}$ pulled back to $\mathbb{S}^*Y$ with values in $\mathcal{G}^0_{b,sc}$, the invertible elements of order zero in the b,sc calculus. It will be useful to have the analogue of the trace defect formula (B.8).

Consider two pure forms with values in $\Psi^0_{b,sc}$, say

$$\eta \in \mathcal{C}^\infty\left(\mathbb{S}^*Y, \Lambda^k\Psi^0_{b,sc}\right), \quad \eta = A\widehat{\eta} \text{ with } \widehat{\eta}\in\Omega^k\mathbb{S}^*Y$$
$$\omega \in \mathcal{C}^\infty\left(\mathbb{S}^*Y, \Lambda^\ell\Psi^0_{b,sc}\right), \quad \omega = B\widehat{\omega} \text{ with } \widehat{\omega}\in\Omega^\ell\mathbb{S}^*Y$$

Notice that

$$\begin{aligned}[\eta,\omega]_s &= \eta\wedge\omega - (-1)^{|\eta|\cdot|\omega|}\,\omega\wedge\eta \\ &= [A,B]\,\widehat{\eta}\wedge\widehat{\omega}.\end{aligned} \tag{C.1}$$

The trace functionals and derivations discussed in Appendix B have natural extensions to forms, e.g.,

$$^R\operatorname{Tr} : \mathcal{C}^\infty\left(\mathbb{S}^*Y, \Lambda^k\Psi^0_{b,sc}\right) \to \Omega^k\mathbb{S}^*Y,$$

by acting on coefficients (thus $^R\operatorname{Tr}(\eta) = {}^R\operatorname{Tr}(A)\,\widehat{\eta}$). The trace defect formulas generalize to this context; in particular

$$\begin{aligned}{}^R\operatorname{Tr}\left([\eta,\omega]_s\right) = -\widehat{\operatorname{Tr}^\sigma}&\left(\frac{D_Q(\eta)\wedge\omega + (-1)^{|\eta|\cdot|\omega|}\,\eta\wedge D_Q(\omega)}{2}\right)\\ &+\widehat{\operatorname{Tr}^\partial}\left(\frac{D_x(\eta)\wedge\omega + (-1)^{|\eta|\cdot|\omega|}\,\eta\wedge D_x(\omega)}{2}\right)\end{aligned} \tag{C.2}$$

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